Mathematics 242 – Principles of Analysis
Final Exam Solutions

I.

A) $A = \{\sin(x) : 0 \le x \le 3\pi/4\} = [0,1]$ and $B = \{x : 1 < x^2 < 4\} = (-2,-1) \cup (1,2)$, so $A \cup B = (-2,-1) \cup [0,2)$.

B) $C = \{|x-2| : x \in A\}$ equals the set $[1,2]$, since we think of $|x-2|$ as the distance from x to 2. Hence $\text{lub}(C) = 2$ (the largest distance to 2 of any element of C).

II.

A) x_n converges to L if for every $\epsilon > 0$, there exists some corresponding $n_0 \in \mathbf{N}$ such that $|x_n - L| < \epsilon$ for all $n \ge n_0$.

B) By the "Big Limit Theorem" (for instance),

$$\begin{aligned}\lim_{n\to\infty} \frac{3n^2+n}{n^2+1} &= \lim_{n\to\infty} \frac{(3n^2+n)\cdot\frac{1}{n^2}}{(n^2+1)\cdot\frac{1}{n^2}} \\ &= \lim_{n\to\infty} \frac{3+\frac{1}{n}}{1+\frac{1}{n^2}} \\ &= 3\end{aligned}$$

To prove this by the definition, let $n \ge 3$. Then

$$\left|\frac{3n^2+n}{n^2+1} - 3\right| = \left|\frac{n-3}{n^2+1}\right| = \frac{n-3}{n^2+1} < \frac{n}{n^2} = \frac{1}{n}$$

Hence, given $\epsilon > 0$, if $n_0 \ge \max\{3, 1/\epsilon\}$ and $n \ge n_0$, then

$$\left|\frac{3n^2+n}{n^2+1} - 3\right| < \frac{1}{n} < \epsilon$$

III.

A) Since $\sin(x)$ takes values only between -1 and 1, x_n is a bounded sequence. Hence, the Bolzano-Weierstrass Theorem implies that x_n has a convergent subsequence x_{n_k}.

B) The answer is *yes*. Using the index sequence n_k found in part A, the sequence y_{n_k} is also a bounded sequence since the cosine function is also bounded. By the Bolzano-Weierstrass Theorem again, there is a convergent subsequence $y_{n_{k_\ell}}$. The corresponding subsequence $x_{n_{k_\ell}}$ is also convergent, since every subsequence of a convergent sequence (like x_{n_k}) is also convergent.

IV.

A) $\lim_{x\to c} f(x) = L$ if for every $\epsilon > 0$ there exists a corresponding $\delta > 0$ such that $|f(x) - L| < \epsilon$ for all x with $0 < |x-c| < \delta$.

B) $\lim_{x\to 0} x^2 \sin(1/x) = 0$ since for all $x \neq 0$,

$$|x^2 \sin(1/x)| = |x|^2 |\sin(1/x)| \leq |x|^2$$

(the last inequality follows since $|\sin(\theta)| \leq 1$ for all θ). Hence, given $\epsilon > 0$, let $\delta = \sqrt{\epsilon}$. Then if $0 < |x| < \delta$,

$$|x^2 \sin(1/x) - 0| \leq |x|^2 < \delta^2 = \epsilon$$

V.

A) The Intermediate Value Theorem: Let f be continuous on $[a, b]$ and let $m = \min\{f(x) : x \in [a, b]\}$, $M = \max\{f(x) : x \in [a, b]\}$. Then for all y_0, $m \leq y_0 \leq M$, there exist $x_0 \in [a, b]$ such that $f(x_0) = y_0$.

B) For $f(x) = a_1 + a_2 \sin(x)$,

$$\begin{cases} f(-\pi/2) = a_1 - a_2 \\ f(\pi/2) = a_1 + a_2 \end{cases}$$

By hypothesis, $a_2 > |a_1|$. Hence $a_1 - a_2 < 0$ and $a_1 + a_2 > 0$. Since f is continuous everywhere, the Intermediate Value Theorem implies that there is some $x_0 \in [-\pi/2, \pi/2]$ such that $f(x_0) = 0$.

VI. The Fundamental Theorem of Calculus: Let f be continuous on $[a, b]$, and let $F(x) = \int_a^x f(t)\, dt$. Then:

1) F is differentiable on $[a, b]$, and $F'(x) = f(x)$.
2) If G is any antiderivative of f, then $\int_a^b f(x)\, dx = G(b) - G(a)$.

See the class notes for the proof.

VII.

A) **False** – f has a jump discontinuity at $x = 1$. If f is differentiable at c it is also continuous at c.

B) **True** – the sequence of partial sums is bounded above and monotone increasing since $a_n \geq 0$ for all n. Hence the sequence of partial sums converges to some real number.

C) **True** – the conclusion of the MVT is that there exists some $c \in (a, b)$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$. This is true here (even though the hypotheses of the MVT are not satisfied!!) since

$$f'\left(\frac{1}{8}\right) = 2 \cdot \frac{1}{8} = \frac{1}{4} = \frac{f(2) - f(0)}{2 - 0}$$

D) **False** – this series is absolutely convergent, not conditionally convergent. The series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^3}$ itself converges by the Alternating Series Test. Then, the series of absolute values:

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{n^3} \right| = \sum_{n=1}^{\infty} \frac{1}{n^3}$$

also converges since it is a p series with $p = 3 > 1$.

E) **True** – We proved in class that if f is continuous on $[a, b]$ except for a finite number of jump or removable discontinuities, then f is integrable. That is the case for f here.

F) **False** – By the Ratio Test,

$$\lim_{n\to\infty} \left| \frac{x^{n+1}}{(n+1)2^{n+1}} \frac{n2^n}{x^n} \right| = \lim_{n\to\infty} \frac{n}{n+1} \frac{|x|}{2} = \frac{|x|}{2}$$

This is > 1 if $|x| > 2$. Hence the series diverges for all x with $|x| > 2$. (It converges absolutely by the Ratio Test for all $|x| < 2$. The series also converges conditionally for $x = -2$, but diverges for $x = 2$.)

G) **True** – This basically comes from the Integral Test calculation that shows the p-series with $p > 1$ is convergent. By thinking of the partial sum $\sum_{n=1}^{N} \frac{1}{n^p}$ as the upper sum (left-hand Riemann sum) for the decreasing function $f(x) = \frac{1}{x^p}$ on $[1, N+1]$ corresponding to the partition $P = \{1, 2, \ldots, N+1\}$, we get

$$\sum_{n=1}^{N} \frac{1}{n^p} \geq \int_1^{N+1} \frac{1}{x^p}\, dx = \frac{(N+1)^{1-p} - 1}{1-p}.$$

As $N \to \infty$, we get

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \geq \lim_{n\to\infty} \frac{(N+1)^{1-p} - 1}{1-p} = \frac{1}{p-1}$$

On the other hand, omitting the first term in the series gives $\sum_{n=2}^{N} \frac{1}{n^p}$ as the lower sum (right-hand Riemann sum) for the decreasing function $f(x) = \frac{1}{x^p}$ on $[1, N]$ corresponding to the partition $P = \{1, 2, \ldots, N\}$. So

$$\sum_{n=2}^{N} \frac{1}{n^p} = \sum_{n=1}^{N} \frac{1}{n^p} - 1 \leq \int_1^{N} \frac{1}{x^p}\, dx = \frac{N^{1-p} - 1}{1-p}$$

Again letting $N \to \infty$, we get

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \leq 1 + \lim_{N\to\infty} \frac{N^{1-p} - 1}{1-p} = 1 + \frac{1}{p-1} = \frac{p}{p-1}$$