Mathematics 242 – Principles of Analysis
Final Examination
December 15, 2000

Directions

Do all work in the blue exam booklet. There are 200 possible regular points and 20 possible Extra Credit points (see problem VII below).

I. Let $A = \{\sin(x) : x \in [0, 3\pi/4]\}$ and let $B = \{x : 1 < x^2 < 4\}$.

A) (10) What is the set $A \cup B$?
B) (10) What is the least upper bound of the set $C = \{|x - 2| : x \in A\}$?

II.

A) (10) State the ϵ - n_0 definition for convergence of a sequence.
B) (20) Identify $L = \lim_{n\to\infty} x_n$ for the sequence

$$x_n = \frac{3n^2 + n}{n^2 + 1}$$

and prove using the definition that $\lim_{n\to\infty} x_n = L$.

III. Let $x_n = \sin(n)$ and $y_n = \cos(n)$. In this problem you may use without proof any facts about sequences, subsequences, etc. that we proved in class.
A) (10) Show that there exists an index sequence of positive integers n_k such that $x_{n_k} = \sin(n_k)$ converges.
B) (10) Does there exist an index sequence n_k of positive integers such that *both* x_{n_k} and y_{n_k} converge? Prove your assertion.

IV.
A) (10) Give the ϵ - δ definition for the statement $\lim_{x\to c} f(x) = L$.
B) (20) Identify the limit

$$L = \lim_{x\to 0} x^2 \sin(1/x)$$

and prove using the definition that $\lim_{x\to 0} x \sin(1/x) = L$.

V.
A) (10) State the Intermediate Value Theorem.
B) (10) Let $f(x) = a_1 + a_2 \sin(x)$, where a_1, a_2 are constants with $|a_1| < a_2$. Show that there exists some $c \in [-\pi/2, \pi/2]$ such that $f(c) = 0$.

VI. (30) State and prove both parts of the Fundamental Theorem of Calculus.

VII. True - False. For each true statement, give a short proof or reason. For each false statement give a reason or a counterexample. Do *any five* of the following. If you submit solutions for more than five parts, correct answers will be counted as Extra Credit. In any parts that refer to $f(x)$, use

$$f(x) = \begin{cases} x^2 & \text{if } x \in [0,1) \\ \frac{1}{2} & \text{if } x \in [1,2] \end{cases}$$

A) (10) $f(x)$ is *differentiable* at $x = 1$.

B) (10) Let $\sum_{n=1}^{\infty} a_n$ be an infinite series with positive terms. If the partial sums s_N are bounded above, then $\sum_{n=1}^{\infty} a_n$ converges.

C) (10) The *conclusion* of the Mean Value Theorem holds for $f(x)$ on the interval $[0, 2]$.

D) (10) The infinite series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^3}$$

converges conditionally.

E) (10) $f(x)$ is *integrable* on $[0, 2]$. (You may use without proof any general theorems about integrability.)

F) (10) The infinite series

$$\sum_{n=1}^{\infty} \frac{x^n}{n2^n}$$

converges for all $x \in \mathbf{R}$.

G) (10) If $p > 1$, then

$$\frac{1}{p-1} \leq \sum_{n=1}^{\infty} \frac{1}{n^p} \leq \frac{p}{p-1}$$

Have a wonderful holiday season!