Mathematics 242 – Principles of Analysis
Selected Solutions – Exam 1
October 11, 2000

I. Let $A = \{x \in \mathbf{R} : 1 < x^2 < 9\}$ and let $B$ be the interval $[-2, 4)$.

A) What is the set $A \cap B$?

*Solution:* $1 < x^2 < 9$ is equivalent to either $1 < x < 3$, or $-3 < x < -1$. So $A$ is the union of two intervals: $A = (-3, -1) \cup (1, 3)$. The intersection with $B$ is

$$A \cap B = [-2, -1) \cup (1, 3)$$

B) Explain why the set $C = \{|x + 1/2| : x \in A\}$ is bounded, and give upper and lower bounds for it.

*Solution:* Since $C$ is a set of numbers obtained by taking absolute values, it is bounded below by 0. Since $x \in A$ implies $|x| < 3$, $|x+1/2| \leq |x|+|1/2| < 3+1/2$ for all $x \in A$. Hence $C$ is also bounded above.

C) What are the *least upper bound* and *greatest lower bound* for the set $C$ from part B? Explain your reasoning.

*Solution:* $|x + 1/2| = |x - (-1/2)|$ is the distance from $x$ to $-1/2$ along the number line. By part A, we know that $A = (-3, -1) \cup (1, 3)$. Hence the closest points to $-1/2$ in $A$ are $x$ near $-1$, and we can get arbitrarily close to $|x + 1/2| = 1/2$ for $x$ close to $-1$ in $A$. The greatest lower bound for $C$ is $1/2$. Similarly, the farthest points from $-1/2$ are the $x$ near 3 in $A$. Hence the least lower bound for $C$ is $7/2$.

II. For parts A, B see the text or class notes. *Comments:* There will be questions like this on the next midterm and final. *You will be throwing away lots of hard-to-replace points if you do not know the definitions, statements of results, and proofs that I put on the review sheet for an exam. Also, part C of this question was the only problem on this exam intended to be slightly "tricky"!*

C) Suppose that $x_n$ is a sequence of *negative* numbers and $x_{n+1}/x_n \leq 1$ for all $n \geq 1$. Show that $x_n$ converges to some $a \in \mathbf{R}$.

*Solution:* Since all the $x_n$ are negative, when we multiply both sides of the inequality $x_{n+1}/x_n \leq 1$ by $x_n$ to clear denominators, the inequality reverses: $x_{n+1} \geq x_n$ for all $n$. Also since all of the $x_n$ are negative, the number $M = 0$ is an upper bound for the sequence. By the result of part B, every increasing sequence that is bounded above converges to some $a \in \mathbf{R}$. The most we can say here is that $a \leq 0$.

III.

A) Identify $\lim_{n\to\infty} x_n$ for the sequence

$$x_n = \frac{3n}{2n+5},$$

and prove that $x_n$ converges to your limit using the $\epsilon$ - $n_0$ definition of convergence.

*Solution:* By our usual methods, the limit should be $\frac{3}{2}$. To prove this, note that

$$\begin{aligned}
\left|\frac{3n}{2n+5} - \frac{3}{2}\right| &= \left|\frac{-15}{4n+10}\right| \\
&= \frac{15}{4n+10} \\
&< \frac{15}{4n}
\end{aligned}$$

Hence, given $\epsilon > 0$, if $n \geq n_0 > \frac{15}{4\epsilon}$, then $\frac{1}{n} < \frac{4\epsilon}{15}$, so we will have

$$\left|\frac{3n}{2n+5} - \frac{3}{2}\right| < \frac{15}{4n} < \frac{15}{4} \cdot \frac{4\epsilon}{15} = \epsilon.$$

By the definition of convergence, this shows that $\lim_{n\to\infty} x_n = \frac{3}{2}$.

B) Use the "Big Limit Theorem" for sequences to determine

$$\lim_{n\to\infty} \frac{\sqrt{4n^2+3n+1}}{\sqrt[3]{27n^3+3}}$$

*Solution:* The fastest growing terms on the top and bottom both are roughly constant times $n$ as $n \to \infty$ (square root of $n^2$ and cube root of $n^3$). Hence, if we divide both top and bottom by $\frac{1}{n}$, we get

$$\begin{aligned}
\lim_{n\to\infty} \frac{\sqrt{4n^2+3n+1}}{\sqrt[3]{27n^3+3}} &= \lim_{n\to\infty} \frac{\sqrt{4n^2+3n+1}\cdot\frac{1}{n}}{\sqrt[3]{27n^3+3}\cdot\frac{1}{n}} \\
&= \lim_{n\to\infty} \frac{\sqrt{4+\frac{3}{n}+\frac{1}{n^2}}}{\sqrt[3]{27+\frac{3}{n^3}}} \\
&= \frac{2}{3}
\end{aligned}$$

C) Consider the sequence

$$x_n = \sin\left(\frac{\pi\sqrt{n}}{2}\right)$$

Find three convergent subsequences of $x_n$ but converging to three different limits. (For each one, give a formula for the index sequence $n_k$ for the subsequence). What does this say about $\lim_{n\to\infty} x_n$?

*Solution:* The simplest way to find the required subsequences is to try to find $n_k \to \infty$ where the sine function takes on some constant value. For instance, we know that $\sin(x) = 0$ for all integer multiples of $\pi$, so if

$$\frac{\pi\sqrt{n}}{2} = k\pi$$

for any integer $k \geq 1$, then we get a zero term in the sequence. From the equation above, $n_k = (2k)^2$ gives one such subsequence, with $x_{n_k} = 0$ for all $k \geq 1$. Hence $\lim_{k\to\infty} x_{n_k} = 0$.
Similarly, to get $\sin(x) = 1$, we can take

$$\frac{\pi\sqrt{n}}{2} = \frac{(4k+1)\pi}{2}$$

for all integers $k \geq 1$. Hence $n_k = (4k+1)^2$ gives a subsequence converging to $+1$.
Finally, to get $\sin(x) = -1$, we can take

$$\frac{\pi\sqrt{n}}{2} = \frac{(4k+3)\pi}{2}$$

for all integers $k \geq 1$. Hence $n_k = (4k+3)^2$ gives a subsequence converging to $-1$.
The fact that we can find these subsequences says that the whole $x_n$ sequence *does not converge.*