Mathematics 242 – Principles of Analysis
Discussion 7 – Infinite Series
November 20 and 27, 2000

Background

The last unit of this course will be devoted to the following general question: How can we tell whether an infinite sum of real numbers

$$(1) \qquad \sum_{k=1}^{\infty} a_k$$

"adds up" or *converges* to a well-defined finite result. A sum with infinitely many terms is called an *infinite series*. The numbers a_k are called the *terms* of the series.

We will approach this question by the following method. The thing that makes sense immediately from the algebraic properties of the real number system is to add a finite collection of numbers. So we can stop after any finite number N of terms in the infinite sum above and consider the "partial sum":

$$s_N = \sum_{k=1}^{N} a_k = a_1 + a_2 + \cdots + a_N$$

In this way, we get a sequence of partial sums, whose behavior should tell us whether the infinite sum (1) converges or not. Of course it is possible for an infinite sum *not* to add up to a finite result. For example, if $a_k = 1$ all k, then $S_N = \frac{N(N+1)}{2}$ which goes to $+\infty$ as $N \to \infty$. We say that the series $\sum_{k=1}^{\infty} 1$ *diverges.* On the other hand, if $\lim_{N\to\infty} s_N$ exists and equals S, then we will say that the series (1) converges to (intuitively, "adds up to") S.

For today, we will concentrate on the case where all $a_k \geq 0$.

Discussion Questions

A) Assuming all $a_k \geq 0$, what can we say about the sequence of partial sums s_N? Under what conditions would that sequence be "guaranteed" to converge to some finite real value?

B) In problem 15 from Section 1.4 you proved that $\sum_{k=1}^{N} \frac{1}{k!} \leq 2$ for all N. What does this say about the infinite series $\sum_{k=1}^{\infty} \frac{1}{k!}$. Experiment with the numerical values for partial sums up to $N = 15$ and see if your statement seems reasonable.

C) In problem 11 from Section 1.4, you proved that if r is any real number, then the finite *geometric series with ratio* r adds up to the following:

$$\sum_{k=0}^{N} r^k = \frac{1 - r^{N+1}}{1 - r}.$$

Suppose $r > 0$. Thinking of the left hand side here as the partial sum for the infinite geometric series $\sum_{k=0}^{\infty} r^k$, when will it be true that this series converges?

D) Consider the series $\sum_{k=1}^{\infty} \frac{1}{k^2+k}$, where $a_k = \frac{1}{k^2+k}$ for all $k \geq 1$. All partial sums of this series are rational numbers. Compute the partial sums $s_1, \ldots, s_8$ and look for a pattern. Prove your guess, and use that to determine an answer to our general question in this case. If $\sum_{n=1}^{\infty} \frac{1}{k^2+k}$ does add up to a finite result, what is the sum?

E) On the other hand, consider the series $\sum_{k=1}^{\infty} \frac{1}{k}$. Consider the partial sums s_2, s_4, s_8, s_{16} and so forth–the partial sums where $N = 2^\ell$ is a power of 2. What can you say about $s_{2^{\ell+1}} - s_{2^\ell}$? Determine an answer to our general question in this case. If $\sum_{n=1}^{\infty} \frac{1}{k}$ does add up to a finite result, what is the sum?

F) Suppose you know that series $\sum_{k=1}^{\infty} a_k$ (all $a_k \geq 0$) *converges*, and that the terms in a second series $\sum_{k=1}^{\infty} b_k$ satisfy $b_k \leq a_k$. How are the Nth partial sums for the a-series and the b-series related? Must the b-series also converge? Prove your assertion.

G) Suppose you know that the $\sum_{k=1}^{\infty} a_k$ (all $a_k \geq 0$) *diverges*, and that the terms in a second series $\sum_{k=1}^{\infty} b_k$ satisfy $b_k \geq a_k$. How are the Nth partial sums for the a-series and the b-series related? Must the b-series also diverge? Prove your assertion.

H) Suppose you know that the $\sum_{k=1}^{\infty} a_k$ (all $a_k \geq 0$) *diverges*, and that the terms in a second series $\sum_{k=1}^{\infty} b_k$ satisfy $b_k \leq a_k$. Can you say anything in this case?

Assignment

Group write-ups due in class Wednesday, November 29.