Mathematics 242 – Principles of Analysis
Discussion 6 – Fixed Point Iteration, Revisited
November 1, 2000

Background

Last time, we proved the "Most Valuable Theorem":

Mean Value Theorem. *Let $f(x)$ be continuous on $[a,b]$ and differentiable on (a,b). Then there exists a $c \in (a,b)$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$.*

We also saw how the MVT establishes connections between properties of $f'(x)$ (such as the sign of the derivative), and properties of the function $f(x)$ (such as whether f is increasing or decreasing). Today, we will see another connection dealing with properties of a function under *iteration*.

We will use the following ideas. Recall that we have seen (Problem Set 4) that if g is continuous and $g : [a,b] \to [a,b]$, then g has a *fixed point* $p \in [a,b]$ – a point satisfying $g(p) = p$. We can attempt to find such a point by starting at some x_0 and iterating g:

$$
\begin{aligned}
x_1 &= g(x_0) \\
x_2 &= g(x_1) \\
x_3 &= g(x_2) \\
&\vdots \\
x_{n+1} &= g(x_n)
\end{aligned}
\tag{1}
$$

Discussion Questions

A) If g is continuous and an iteration sequence x_n, constructed using g as in (1) above converges to some $q \in \mathbf{R}$, must q satisfy $g(q) = q$? Why or why not?

B) Try the following choices for g and x_0. Using a calculator if necessary, determine whether it appears that the sequence x_n constructed as in (1) above is converging to some $q \in \mathbf{R}$.

1) $g(x) = \cos(x)$, $x_0 = 0$.
2) $g(x) = \frac{x^2+5}{2x}$, $x_0 = 1$.
3) $g(x) = (x^3 + 2x^2 - x - 2)/10$, $x_0 = 1$ and $x_0 = 3$.

C) For each of the functions in question B, plot $y = g(x)$ together with the line $y = x$. How many fixed points does each function have? Where are they, approximately, and how do they correlate with the points you found in question B?

You can visualize an iteration sequence this way: Draw the graph $y = g(x)$ together with the line $y = x$. Starting from $x = x_0$ (on the x-axis), draw a vertical line segment up or down to the point $(x_0, g(x_0))$ on the graph $y = g(x)$. Then connect that point to the point on the line $y = x$ with the same y-coordinate. The vertical line through that point intersects the x-axis in $x_1 = g(x_0)$ (why?). Then repeat the process to get $x_2 = g(x_1)$, etc.

D) Draw diagrams using this idea to illustrate the iteration sequences from question B. Do the same for iteration sequences for $g(x) = x^3 + 2x^2 - x - 2$ (10 times the polynomial from B, part 3), with $x_0 = 0$ and $x_0 = 3$.

E) From the examples you have seen, given a function $g(x)$ and a fixed point q of g, when does it appear that iteration starting from x_0 close to q will yield a sequence x_n converging to q?

F) If we assume that $g(x)$ is differentiable on an interval containing a fixed point q, the Mean Value Theorem can be used to prove a statement guaranteeing convergence to the fixed point. Think about the following question: Given $q = g(q)$ and x_0 near q, if $x_1 = g(x_0)$, then by the MVT,

$$|x_1 - q| = |g(x_0) - g(q)| = |g'(c)||x_0 - q|$$

for some $c \in (x_0, q)$. When is it guaranteed that x_1 is *closer to q than x_0 was?* Then consider subsequent steps in the iteration.

Assignment

Group write-ups, due Monday, November 6.