Mathematics 242 – Principles of Analysis
Discussion 5 – Between the Max and the Min
October 20, 2000

Background

Last time, we saw the proof of the Extreme Value Theorem. If $f : [a,b] \to \mathbf{R}$ is a function that is continuous at every point x in the closed interval $[a,b]$, then f attains its maximum and and minimum values on $[a,b]$ – that is:

a) The set $S = \{f(x) : x \in [a,b]\}$ is bounded above and below in $\mathbf{R}$,
b) By completeness of $\mathbf{R}$, S has a least upper bound, M, and a greatest lower bound, m, AND
c) There exists some $x_1 \in [a,b]$, such that $f(x_1) = M$. Similarly, there is some $x_2 \in [a,b]$ such that $f(x_2) = m$.

Today, we will ask: What happens for $y \in \mathbf{R}$ *between* m and M – for y in the range $m \leq y \leq M$?

Discussion Questions

A) Use our intuitive notion of continuity (the graph $y = f(x)$ for $x \in [a,b]$ has no "holes, breaks, or jumps") to formulate a response to the following question: Is it necessarily that case that for each $y \in [m, M]$, there exists some (at least one) $x \in [a,b]$ such that $f(x) = y$?

B) Can you prove your conjecture? One suggestion: Suppose you try something like our proof of the Bolzano-Weierstrass Theorem, attempting to "zero in" on a solution of $f(x) = y$. Consider the midpoint $x_1 = \frac{a+b}{2}$ of $[a,b]$. If $f(x_1) = y$, then of course, you are done! But if not, then f has a max M_1 and a min m_1 on $[a, x_1]$, and a max M_1' and a min m_1' on $[x_1, b]$. Can you discard one half in such a way that y is still between the max and the min on the half you *keep*? Now repeat the process on the remaining interval (forever) to generate a nested sequence of intervals:

$$[a,b] = I_2 \supset I_2 \supset I_3 \supset I_4 \supset \cdots \supset I_n \supset \cdots$$

What happens in the limit as $n \to \infty$?

C) Suppose $f(x)$ is a polynomial of *odd degree*:

$$f(x) = c_{2k+1}x^{2k+1} + c_{2k}x^{2k} + \cdots + c_1 x + x_0$$

($c_{2k+1} \neq 0$). We know that $f(x)$ is continuous at all $x \in \mathbf{R}$. Show that the equation $f(x) = 0$ must have at least one solution $x \in \mathbf{R}$.

Assignment

Group write-ups due Wednesday, October 25.