Mathematics 242 – Principles of Analysis
Discussion 4 – Limits for $f(x)$, a First Approach
September 29, 2000

Background

Our next goal is to study precisely what it should mean to say that a real-valued function of a real variable x has a limit as $x \to a$. That is we want to understand what should it mean to say that $\lim_{x\to a} f(x) = L$ for some real number L. (This will be used to define the derivative of a function so it's really the basis for calculus.) For the moment, we will have to rely on the *intuitive* idea of limits of functions that you probably used in your calculus courses. Namely, saying that $\lim_{x\to a} f(x) = L$ should mean roughly that the values of the function $f(x)$ are approaching L as x approaches a (but $x \neq a$). You should be aware, though, that you have not seen a very broad class of examples of functions, so your intuition may not be too reliable(!)

Fortunately, we do know a lot about limits of sequences now, so it is natural to try to use this here to capture the $x \to a$ part of the limit idea. Today we will try to determine exactly what the connection should be. Note that if we have a sequence $x_n \to a$ and all the x_n are in the domain of the function, then we can substitute these x-values into f, and get a corresponding sequence of function values $f(x_n)$, and we can ask what the sequence $f(x_n)$ is doing as $n \to \infty$.

Discussion Questions

A) Consider the function $f(x) = x^2 + 2$ and $a = 3$.

1) Based on the intuitive idea, should $\lim_{x\to 3} f(x)$ exist? What should it be?
2) Let $x_n = 3+\frac{1}{n}$, a sequence converging to $a = 3$. What can you say about $\lim_{n\to\infty} f(x_n)$? (Use the "Big Limit Theorem" here and whenever possible in the following. It is not necessary to use the $\epsilon - n_0$ definition of convergence for sequences.)
3) Same question for $y_n = \frac{3n^2-1}{n^2+1} \to 3$.
4) (You'll need a calculator for this one.) Experiment with the sequence $z_n = 9n\sin\left(\frac{1}{3n}\right)$. What does it seem to be doing as $n \to \infty$? What does the sequence $f(z_n)$ seem to be doing as $n \to \infty$?

B) Now consider the function

$$f(x) = \begin{cases} x^2 & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$$

1) Based on the intuitive idea, should $\lim_{x\to 0} f(x)$ exist? What should it be?
2) Take the sequence $x_n = \frac{(-1)^n}{n^2} \to 0$. Does $f(x_n)$ have a limit as $n \to \infty$?
3) What about $y_n = \frac{10^n}{n!} \to 0$? What can you say about $f(y_n)$ as $n \to \infty$? (You will probably want to refer back to Discussion 3, question B here.)

4) (Careful, careful!) Consider the sequence $z_n = \frac{(1+(-1)^n)}{2^n}$. What does this sequence do as $n \to \infty$? What does $f(z_n)$ do as $n \to \infty$?

C) Now consider the following function:

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is an irrational number} \\ \frac{1}{p} & \text{if } x = \frac{m}{p^k} \text{ in lowest terms, } p \text{ prime} \\ \frac{1}{m} & \text{if } x = \frac{m}{n} \text{ is rational, but } n \text{ not a prime power} \end{cases}$$

For instance,

$$\begin{cases} f(\sqrt{2}) = 0 & \text{since } \sqrt{2} \text{ is irrational} \\ \\ f(\frac{3}{64}) = \frac{1}{2} & \text{since } 64 = 2^6 \text{ is a power of a prime} \\ \\ f(\frac{17}{100}) = \frac{1}{17} & \text{since } 100 = 2^2 5^2 \text{ is not a power of a prime} \end{cases}$$

It's not so easy to apply the intuitive idea in a case like this, so let's try to use some sequences converging to 0 to try to see whether $\lim_{x\to 0} f(x)$ should exist.

1) Consider the sequence $x_n = \frac{\sqrt{2}}{5^n} \to 0$. Does $f(x_n)$ have a limit as $n \to \infty$?
2) What about $f(y_n)$ for $y_n = \frac{1}{7^n} \to 0$?
3) What about $f(z_n)$ for $z_n = \frac{(2+(-1)^n)}{n} \to 0$?
4) What do these examples tell you about the possibility of identifying a real number L such that $\lim_{x\to 0} f(x) = L$ for this function?

D) Putting it all together. Write a paragraph addressing the following questions: If you want to try to determine whether $\lim_{x\to a} f(x) = L$, how can you use sequences $x_n \to a$? When should it be true that $\lim_{x\to a} f(x)$ exists and equals L? (Be careful here; the example in B, part 4 is important!) When would $\lim_{x\to a} f(x)$ not exist? Explain how each part of the previous three questions fits into what you are saying.

Assignment

Group writeups, due in class Wednesday, October 4.