Mathematics 242 – Principles of Analysis
Discussion 3 – Using the "Big Limit Theorem"
September 20, 2000

Background

Last time we proved a "Big Limit Theorem" showing how convergent sequences can be combined to produce other convergent sequences using the algebraic sum and product operations on $\mathbf{R}$.

Theorem. *Let $x_n \to L$ and $y_n \to M$ as $n \to \infty$. Then*
a) $x_n \pm y_n \to L \pm M$,
b) $x_n \cdot y_n \to L \cdot M$, and
c) if $M \neq 0$, and $y_n \neq 0$ for all $n \in \mathbf{N}$, then $\frac{x_n}{y_n} \to \frac{L}{M}$.

It is also true, for instance, that if $x_n \geq 0$ for all n and $x_n \to L$, then $\sqrt{x_n} \to \sqrt{L}$ as $n \to \infty$.

One of the very nice features of these results is how they give *justifications* for the "intuitive" approaches we have used to determine limits (e.g. by rearranging the formula for the nth term in various algebraic ways). In this discussion, we will see how that works, then look at some additional facts that can be derived from the "Big Limit Theorem".

Discussion Questions

A) For each of the following convergent sequences, use one or more parts of the "Big Limit Theorem" to find the limit, and explain how you are using the theorem. You may use without proof "standard facts" such as $\lim_{n\to\infty} \frac{1}{n} = 0$, or $\lim_{n\to\infty} c^n = 0$ if $|c| < 1$.

1) $x_n = \frac{4n^3+2}{n^3+n}$
2) $y_n = \frac{2^n}{1.5^n+2.5^n}$ (Hint: Try to identify the fastest growing term, then rearrange algebraically to "neutralize" it.)
3) $z_n = \sqrt{4n^2+n} - \sqrt{4n^2-3}$ (Hint: The algebraic fact $(\sqrt{a}-\sqrt{b})(\sqrt{a}+\sqrt{b}) = a-b$ may be useful here.)

B) In this problem you will develop a useful sufficient condition for when $x_n \to 0$ based on examining the *ratios* between successive terms: $\frac{x_{n+1}}{x_n}$.

1) Assume that x_n is a sequence of positive numbers and that for a fixed $c < 1$, $\frac{x_{n+1}}{x_n} < c$ for all $n \geq 1$. Show (using one or more parts of the "Big Limit Theorem") that $\lim_{n\to\infty} x_n = 0$.
2) Now assume that x_n is a sequence of positive numbers such that $\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = a < 1$. Show that again $\lim_{n\to\infty} x_n = 0$. (Hint: If z_n is a sequence with $\lim_{n\to\infty} z_n = a < 1$ and $a < c < 1$, then there exists an $n_0 \in \mathbf{N}$ such that $z_n < c$ for all $n \geq n_0$. The idea is the same as in problem 6 from §2.1.)

3) Use part 2 of this question to show that

$$\lim_{n\to\infty} \frac{b^n}{n!} = 0.$$

for all $b \in \mathbf{R}$.

4) Use part 2 of this question to show that

$$\lim_{n\to\infty} \frac{(n!)^2}{(2n)!} = 0.$$

Assignment

Group writeups, due in class Monday, September 25.