Mathematics 242 – Principles of Analysis
Discussion 2 – Completeness of $\mathbf{R}$, continued
September 8, 2000

Background

In class on Wednesday, we introduced the *L.U.B., or Completeness Axiom* for the real number system:

Completeness (LUB) Axiom. *Every nonempty set of real numbers that is bounded above has a least upper bound (lub) in the real numbers.*

Today, we want to begin to explore some of the consequences of this innocuous-seeming but *extremely important* property.

Discussion Questions

A) Identify a lub for each of the following nonempty sets of real numbers. You may use facts you know from calculus if needed.

1) $A = \{x \mid x^2 - 3x - 6 \leq 0\}$
2) $B = \left\{\frac{2n}{n+3} \mid n \in \mathbf{N}\right\}$
3) $C = \{\cos x \mid 0 < x \leq \pi/2\}$

B) In this problem, you will show that if A is a set of real numbers that is bounded above and nonempty, then $\text{lub}(A)$ is unique – we are actually justified in saying *"the least upper bound"* instead of *"a least upper bound"*. To show that something is unique, a standard approach is to assume you have any two of them, then show that the two must be equal. So suppose that α and α' are both least upper bounds of A. Then ... (you complete the proof). (*Hint*: Go back to our original definition of a lub and use what that says.)

C) For sets that are bounded *below* there is a corresponding notion of *greatest lower bound.*

1) Give a precise definition of what it should mean for $\beta \in \mathbf{R}$ to be a greatest lower bound for a set B that is bounded below.
2) Prove that the Completeness Axiom above implies that every nonempty subset of the real numbers that is bounded below has a greatest lower bound. (*Hint: The best way to think about this is that you can deduce what you want from the LUB axiom if you can "turn lower bounds into upper bounds" somehow – for instance by doing some operation to your set that reverses all inequalities.*)

D) For each of the following statements that is true, give a short proof; for those that are false, give a counterexample.

A) Suppose $\alpha = \text{lub}(A)$. Then $|\alpha| = \text{lub}(\{|x| | x \in A\})$.
B) Suppose $\alpha = \text{lub}(A)$ and $\beta = \text{lub}(B)$. Then $\alpha + \beta = \text{lub}(\{x + y | x \in A, y \in B\}$.

Assignment

Group writeups due in class, Wednesday, September 13.