Mathematics 242 – Principles of Analysis
Discussion 1: Rational and Irrational Numbers
September 1, 2000

Background

The purpose of today's discussion is to start to think about what properties *number systems* should have in order for us to be able to solve equations, analyze functions, and ultimately "do calculus." You have used the set of *rational numbers* or fractions:

$$\mathbf{Q} = \{m/n : m, n \text{ integers}, n \neq 0\}$$

since grade school. This collection of numbers is sufficient (for example) for solving simple linear equations of the form:

$$ax + b = c$$

with $a, b, c \in \mathbf{Q}$. We can solve for x, and obtain a rational number as the result:

(1) $$ax + b = c \text{ implies } x = \frac{c}{a} + \frac{(-b)}{a}$$

Discussion Questions

A)

1) What algebraic properties of the integers and rational numbers are we using to solve the equation in (1) and write the solution in this form?
2) Show carefully that if $a = m/n$, $b = p/q$, $c = r/s$, with m, n, p, q, r, s all integers with $n, p, s \neq 0$, then the solution in (1) can also be written as a quotient of integers with nonzero denominator. Give a reason why every step in your algebraic manipulations is valid, using known properties of fractions.
3) How does what you did in parts 1 and 2 relate to the first five axioms for the real numbers on page 13 of the text? (Note: an "axiom" is a property that is assumed to be true, without proof, as part of the description or definition of a mathematical object.)

B) In this and the next question, we will "move up" to quadratic equations. Some quadratic equations with rational number coefficients can also be solved in the rational numbers. For instance, $x^2 - 5x + 6 = 0$ has solutions $x = 2$ and $x = 3$. We want to consider a different equation, though: $x^2 - 2 = 0$. From algebra, you know the solutions $x = \pm\sqrt{2}$. The question is: Are these numbers elements of the set $\mathbf{Q}$?

1) You probably know a way to tell whether a number is rational by looking at its *decimal expansion*. What is special about the decimal expansion of a rational number? (Note: The way to prove this fact is usually discussed in the Algebraic Structures class.)

2) But can you tell anything by looking at a finite part of the decimal expansion $\sqrt{2} = 1.4142136\cdots$? (Note: a finite part is all a human being could ever see!)
3) Here's a different way to try to answer our question. Suppose $\sqrt{2}$ is in $\mathbf{Q}$, and write $\sqrt{2} = m/n > 0$, with positive integers m, n. We can assume that the fraction m/n is in lowest terms (no common factors between m and n). Then

$$(m/n)^2 = 2, \quad \text{or} \quad m^2 = 2n^2.$$

Can you find integers m, n with no common factors satisfying this equation? Why or why not? (Think about the *factorizations* of m, n, then m^2 and n^2 into powers of prime numbers.)

C) Here's another way to get approximations to $\sqrt{2}$. Note that $\sqrt{2}$ solves the equation $x = 2/x$.

1) Explain why for all positive x_0, $\sqrt{2}$ lies between x_0 and $2/x_0$. (Sketching the graphs $y = x, y = 2/x$ on the same set of axes may be useful!)

So, thinking of x_0 as a first approximation, the *average*

$$x_1 = (1/2)(x_0 + 2/x_0)$$

is probably a *better* approximation to $\sqrt{2}$ than x_0 was. Similarly, $\sqrt{2}$ is between x_1 and $2/x_1$, so $x_2 = (1/2)(x_1 + 2/x_1)$ is probably closer than x_1 was. We can repeat this as many times as we like to make

(2) $$x_{n+1} = (1/2)(x_n + 2/x_n)$$

for $n = 0, 1, 2, 3, \ldots$.

2) Using (1), construct the $x_1, \ldots, x_5$ starting with $x_0 = 1$, then restart from $x_0 = 3$ and find $x_1, \ldots, x_5$. What seems to be happening in either case?
3) Prove that all x_n constructed with (1) will be rational numbers if x_0 is a rational number.
4) If you assume that the x_n are approaching some fixed number a as n gets larger and larger, from (2), what number does a have to be?
5) What properties of rational and real numbers have you used in this question?

Writeups: Group writeups due in class Wednesday, September 6.