Mathematics 304, Section 2 – Ordinary Differential Equations
Final Examination – December 16, 2004

Directions: Do all work in the blue exam booklet. You may use a calculator and the table of integrals at any point. There are 200 total regular points and 20 possible Extra Credit points.

I. Consider the first order ODE

$$y' = \cos(t)(y-2)^{\frac{1}{3}} \tag{1}$$

A) (15) Find a solution of the initial value problem for (1) with $y(0) = 3$.

B) (15) What does the Existence and Uniqueness Theorem say about solutions of the initial value problem for (1) with $y(0) = 2$? Show that the constant function $y(t) \equiv 2$ is one solution. Is it the only one?

II. (25) Sketch the bifurcation diagram for the family of 1st order ODE given by $x' = x^2 - a$, where $a \in \mathbf{R}$ is an arbitrary real parameter. In your sketch, show the equilibrium points for the different equations in the family, and identify their types.

III. The temperature y of an insulated shed with no internal heating or cooling varies according to the following ODE (from Newton's Law of Cooling):

$$y' = (1.5)(A(t) - y). \tag{2}$$

In this equation, $A(t)$ is the outside temperature, which varies as a scaled and shifted cosine function with a minimum of 40° F at 12 midnight and a maximum of 70° F at 12 noon.

A) (5) Letting $t = 0$ correspond to 12 noon, write a formula for $A(t)$.

B) (15) Using your answer from part A, solve (2) for $y(t)$.

C) (5) Your answer from part B should have an arbitrary constant. Does the value of that constant in a particular solution affect the *long-term behavior* of y? Explain.

IV. Consider the family of first order systems:

$$X' = \begin{pmatrix} a & 4a \\ 1 & 0 \end{pmatrix} X,$$

where a is a real parameter.

A) (5) For what range of a-values will this system have a *spiral sink* at $(0,0)$?

B) (15) Find the general solution and sketch the phase portrait of the system in this family with $a = 2$.

V. (20) Determine the general solution of the first order system

$$X' = \begin{pmatrix} 2 & 0 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & -3 \end{pmatrix} X$$

using the canonical form of the coefficient matrix and change of basis.

VI. Consider the following 2nd order ODE.

$$y'' + 6y' + 9y = 3e^{-t} + t^2 \tag{3}$$

A) (10) Find the general solution of the associated homogeneous equation.

B) (10) Find a particular solution of (3) by the method of undetermined coefficients.

C) (10) How would the form of the solutions of (3) change if the coefficient of the y term was increased to $9 + \varepsilon$ for $\varepsilon > 0$?

VII. All parts of the following problem refer to the 1st order system

$$\begin{aligned} x' &= -y + x + 1 \\ y' &= x(x - 2y + 1). \end{aligned}$$

A) (10) Find all the equilibrium points.

B) (15) Compute the linearization at each critical point and use that information to determine the type of each (sink, source, saddle, or center).

VIII.

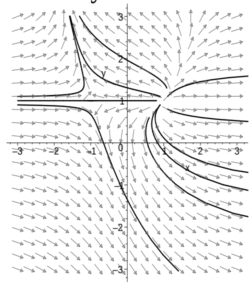
A) (5) Is the following first order system Hamiltonian? Why or why not?

$$\begin{aligned} x' &= xe^{xy} \\ y' &= -ye^{xy} \end{aligned}$$

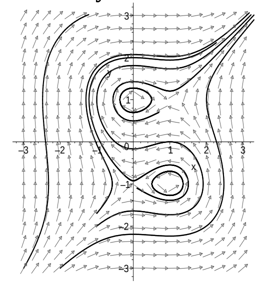
B) (10) Show that the linearization of a Hamiltonian system at an equilibrium always has the form $V' = AV$, where $\mathrm{Tr}(A) = 0$.

C) (10) Which of the following phase portraits could show Hamiltonian systems? (There could be several.)

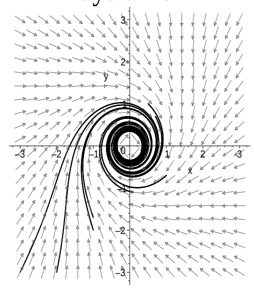
System 1



System 2



System 3



Extra Credit. A non-linear system

$$\begin{aligned} x' &= f(x,y) \\ y' &= g(x,y) \end{aligned}$$

is said to be a *gradient system* on $U \subset \mathbf{R}^2$ if there is some function $G(x,y)$ such that $f(x,y) = \frac{\partial G}{\partial x}$ and $g(x,y) = \frac{\partial G}{\partial y}$ at all points in U. (This is the same as saying the vector field $(f,g) = \nabla G$, the gradient vector field for the function G, hence the name.)

A) (10) Show that the value of G increases along solution curves of the gradient system, and deduce that a gradient system never has nonconstant periodic solutions.

B) (10) What can you say about equilibrium points of gradient systems?

Have a peaceful and joyous holiday season!