Mathematics 304 – Ordinary Differential Equations
Selected Solutions – Problem Set 3
October 25, 2000

§4.5/ 54.

a) The auxiliary equation of $y'' - s^2 y = 0$ is $0 = r^2 - s^2 = (r - s)(r + s)$, so by results proved in class, a fundamental set of solutions is $\{y_1, y_2\} = \{e^{sx}, e^{-sx}\}$, and the general solution looks like
$$y = c_1 e^{sx} + c_2 e^{-sx}$$
where c_1, c_2 are arbitrary constants.

b) Each function $c_1 e^{sx} + c_2 e^{-sx}$ can also be written as
$$\begin{aligned} c_1 e^{sx} + c_2 e^{-sx} &= c_1(\cosh(sx) + \sinh(sx)) + c_2(\cosh(sx) - \sinh(sx)) \\ &= (c_1 + c_2)\cosh(sx) + (c_1 - c_2)\sinh(sx) \end{aligned}$$
Conversely, every linear combination of $\cosh(sx), \sinh(sx)$ can also be rewritten as:
$$\begin{aligned} d_1 \cosh(sx) + d_2 \sinh(sx) &= d_1 \frac{e^{sx} + e^{-sx}}{2} + d_2 \frac{e^{sx} - e^{-sx}}{2} \\ &= \left(\frac{d_1 + d_2}{2}\right) e^{sx} + \left(\frac{d_1 - d_2}{2}\right) e^{-sx} \end{aligned}$$
Therefore $d_1 \cosh(sx) + d_2 \sinh(sx)$ is also a general solution of $y'' - s^2 y = 0$, since this form produces exactly the same collection of functions as the form $c_1 e^{sx} + c_2 e^{-sx}$ when c_i and d_i take on all possible real values.

c) To solve inital value problem for $y'' - s^2 y = 0$ with $y(0) = a$, $y'(0) = b$, we can use either form. The second form of the general solution, $d_1 \cosh(sx) + d_2 \sinh(sx)$, is somewhat simpler. Note that $\cosh(0) = 1$ and $\sinh(0) = 0$ by definition. Hence from $a = y(0) = d_1 \cosh(0) + d_2 \sinh(0) = d_1$, we get $d_1 = a$ immediately. Then $y'(0) = sd_1 \sinh(0) + sd_2 \cosh(0) = sd_2$ so the other initial condition gives $b = y'(0) = sd_2$, so $d_2 = b/s$:
$$y(x) = a\cosh(sx) + \frac{b}{s}\sinh(sx)$$

§4.2/37. a) We assume that $L(y) = y'' + py' + qy = 0$ has p, q continuous on (a,b), and that ϕ is a solution of this ODE that *is not identically zero*, but has a non-isolated zero x_0. In other words, every deleted neighborhood of x_0 contains another zero of ϕ: For all $\delta > 0$, there exists some $x \neq x_0$ with $|x - x_0| < \delta$ such that $\phi(x) = 0$. In particular for $\delta = 1/n$, we get an x_n with $|x_n - x_0| < 1/n$ such that $\phi(x_n) = 0$. By the construction, the x_n form a sequence of real numbers converging to x_0. Now, in the definition of the derivative $\phi'(x_0)$:
$$\phi'(x_0) = \lim_{x \to x_0} \frac{\phi(x) - \phi(x_0)}{x - x_0},$$

we can let $x \to x_0$ through the terms in the x_n sequence. Since we require that $\phi'(x_0)$ exists, we should get the same limit no matter how $x \to x_0$. But $\phi(x_n) = 0$ for all n, so

$$\phi'(x_0) = \lim_{n\to\infty} \frac{\phi(x_n) - \phi(x_0)}{x_n - x_0} = \lim_{n\to\infty} \frac{0}{x_n - x_0} = 0.$$

Therefore $L(\phi) = 0$, $\phi(x_0) = 0$, and $\phi'(x_0) = 0$.

b) The *zero function* also satisfies the IVP $L(y) = 0$, $y(x_0) = 0$, and $y'(x_0) = 0$. Therefore, since we assume p, q are continuous on an interval containing x_0, the Existence and Uniqueness Theorem (Theorem 2, page 165 of text) applies. Hence ϕ from part a) must be the zero function. But this is a contradiction, since we assumed that ϕ was *not identically zero*. Therefore, no solution of $L(y) = 0$ can have a non-isolated zero.

§4.3/4. We are given $y_1 = x^2 \cos(\ln(x))$, $y_2 = x^2 \sin(\ln(x))$. To decide whether these are linearly independent or not, we can consider a linear combination $c_1 y_1(x) + c_2 y_2(x) \equiv 0$ (i.e. the linear combination adds up to the zero function). Then we get another true equation by substituting any $x \in (0, 1)$ in to this equation. For instance with $x = e^{-\pi}$, we get

$$\begin{aligned} 0 &= c_1 y_1(e^{-\pi}) + c_2 y_2(e^{-\pi}) \\ &= c_1 e^{-2\pi} \cos(-\pi) + c_2 e^{-2\pi} \sin(-\pi) \\ &= -c_1 e^{-2\pi} \end{aligned}$$

(since $\sin(-\pi) = 0$). Hence $c_1 = 0$. Then substituting $x = e^{-\pi/2}$, we get $c_2 = 0$ as well. $\{y_1, y_2\}$ is *linearly independent.* The Wronskian works out to

$$W[y_1, y_2](x) = x^3$$

(using $\cos^2(\ln(x)) + \sin^2(\ln(x)) = 1$).

23. If $y = ue^{-\frac{1}{2}\int p(x)\ dx}$, then by the Chain and Product Rules for derivatives

$$\begin{aligned} \frac{dy}{dx} &= ue^{-\frac{1}{2}\int p(x)\ dx} \left(\frac{-1}{2}p(x)\right) + \frac{du}{dx} e^{-\frac{1}{2}\int p(x)\ dx} \\ &= e^{-\frac{1}{2}\int p(x)\ dx} \left(\frac{du}{dx} + \frac{-1}{2}p(x)\right) \end{aligned}$$

Continuing to compute the second derivative, again by Chain and Product Rules:

$$\frac{d^2y}{dx^2} = e^{-\frac{1}{2}\int p(x)\ dx} \left(\frac{d^2u}{dx^2} - p\frac{du}{dx} - \frac{1}{2}\frac{dp}{dx}u + \frac{1}{4}p^2 u\right)$$

We substitute these into

$$0 = L(y) = \frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y$$

and factor out the $e^{-\frac{1}{2}\int p(x)\,dx}$ from all terms. *The* $-p\frac{du}{dx}$ *from* $\frac{d^2y}{dx^2}$ *and the* $p\frac{du}{dx}$ *from* $p\frac{dy}{dx}$ *cancel* leaving

$$0 = e^{-\frac{1}{2}\int p(x)\,dx}\left(\frac{d^2u}{dx^2} + \left(\frac{-1}{2}p' - \frac{1}{4}p^2 + q\right)u\right)$$

Multiplying by $e^{\frac{1}{2}\int p(x)\,dx}$, we get a second order ODE of the desired form for the function u:

$$0 = \frac{d^2u}{dx^2} + \left(\frac{-1}{2}p' - \frac{1}{4}p^2 + q\right)u$$

where $f(x) = \frac{-1}{2}p'(x) - \frac{1}{4}p(x)^2 + q(x)$.

31. We have $L(y) = x^2y'' - 3xy' + 3y = 0$ (a Cauchy-Euler equation).

a) $y_1 = x$ and $y_2 = x^3$ are solutions on $(-\infty, \infty)$ since

$$\begin{aligned} L(x) &= x^2 \cdot 0 - 3x \cdot 1 + 3x = 0 \\ L(x^3) &= x^2 \cdot 6x - 3x \cdot 3x^2 + 3x^3 = 0 \end{aligned}$$

The Wronskian of y_1 and y_2 is

$$W[y_1, y_2](x) = \det\begin{pmatrix} x & x^3 \\ 1 & 3x^2 \end{pmatrix} = 2x^3$$

This is not zero (e.g.) at $x = 1$.

b) $\{y_1, y_2\}$ is linearly independent, since if $c_1y_1 + c_2y_2 = c_1x + c_2x^3 \equiv 0$, then we would get true equations by substituting any value for x. E.g., with $x = 1$ and $x = 2$, we get the equations

$$\begin{aligned} 0 &= c_1 + c_2 \\ 0 &= 2c_1 + 8c_2 \end{aligned}$$

The only solution is $c_1 = c_2 = 0$, so these are linearly independent.

c) $y_3 = |x|^3$ is also a solution of $L(y) = 0$ on $(-\infty, \infty)$ since

$$y_3(x) = \begin{cases} x^3 & \text{if } x \geq 0 \\ -x^3 & \text{if } x < 0 \end{cases}; \qquad y_3'(x) = \begin{cases} 3x^2 & \text{if } x \geq 0 \\ -3x^2 & \text{if } x < 0 \end{cases}; \qquad y_3''(x) = \begin{cases} 6x & \text{if } x \geq 0 \\ -6x & \text{if } x < 0 \end{cases}$$

Therefore

$$L(y_3) = \begin{cases} x^2 \cdot 6x - 3x \cdot 3x^2 + 3x^3 = 0 & \text{if } x \geq 0 \\ x^2 \cdot (-6x) - 3x \cdot (-3x^2) - 3x^3 = 0 & \text{if } x < 0 \end{cases}$$

so y_3 is also a solution defined on $(-\infty, \infty)$.

d) Is $y_3 = c_1y_1 + c_2y_2$ for some c_1, c_2? The answer is no: Let $x = 1$; we get $1 = c_1 + c_2$. But with $x = -1$ we get $1 = -c_1 - c_2$. These equations are inconsistent. So there are no $c_1, c_2 \in \mathbf{R}$ such that $y_3 = c_1y_1 + c_2y_2$ for all $x \in \mathbf{R}$. (Note that there are such constants, though, if we only take $x > 0$ or only $x < 0$.)

e) We are *not* seeing a contradiction to the uniqueness of solutions here since the theorem on existence and uniqueness of solutions *does not apply* on the interval $(-\infty, \infty)$. To write our equation in the standard form $y'' + py' + qy = 0$, we would have to divide through by x^2:

$$y'' + \left(\frac{-3}{x}\right) y' + \left(\frac{3}{x^2}\right) y = 0$$

But then the $p = \frac{-3}{x}$ and $q = \frac{3}{x^2}$ functions have discontinuities at $x = 0$. The fact that there is not a unique solution with $y(1) = 1, y'(1) = 3$ is not a contradiction!