Mathematics 304 – Ordinary Differential Equations
Selected Solutions – Problem Set 2
September 25, 2000

§2.6/35. To solve:

$$(3t - x - 6)dt + (t + x + 2)dx = 0. \tag{1}$$

As discussed in the text, for 1st order equations $Mdt + Ndx = 0$ where $M$ and $N$ are both linear in $x, t$, if we first translate the origin, we can remove the constant terms. Here let

$$\begin{cases} t = u + h \\ x = v + k \end{cases}$$

where $h, k$ are constants. We get from (1):

$$(3u - v + (3h - k - 6))du + (u + v + (h + k + 2))dv = 0 \tag{2}$$

We want to choose $h, k$ to satisfy

$$\begin{cases} 3h - k - 6 = 0 \\ h + k + 2 = 0, \end{cases}$$

so solving simultaneously, $h = 1$ and $k = -3$. From (2) we get the following ODE, *thinking of $u$ as the independent variable*:

$$\frac{dv}{du} = \frac{v - 3u}{v + u} = \frac{\frac{v}{u} - 3}{\frac{v}{u} + 1}. \tag{3}$$

This is a *homogeneous first order equation*, so we solve it by changing dependent variables, letting $z = \frac{v}{u}$, or $v = uz$, so

$$\frac{dv}{du} = z + u\frac{dz}{u}.$$

Substituting in (3) and simplifying we get:

$$\begin{aligned} z + u\frac{dz}{du} &= \frac{z-3}{z+1}, \text{ so} \\ u\frac{dz}{du} &= \frac{z-3}{z+1} - z \\ &= \frac{-z^2 - z + z - 3}{z+1} \\ &= -\frac{z^2+3}{z+1} \end{aligned}$$

This is a *separable* equation; after separating variables, we get:

$$\int \frac{z+1}{z^2+3}dz = -\int \frac{1}{u}du \tag{4}$$

The integral on the left can be separated into two terms:

$$\int \frac{z}{z^2+3} dz + \int \frac{1}{z^2+3} dz$$

In the first one, letting $s = z^2+3$, we see that the form is $\frac{1}{2}\int \frac{1}{s} ds = \frac{1}{2}\ln|s| = \frac{1}{2}\ln(z^2+3)$. The second integral can be obtained from the table in the endpapers of the text (it is an arctangent form): $\int \frac{1}{z^2+3} dz = \frac{1}{\sqrt{3}} \arctan\left(\frac{z}{\sqrt{3}}\right)$. The right side of (4) is the usual logarithm integral. Hence from (4) we get

$$\frac{1}{2}\ln(z^2+3) + \frac{1}{\sqrt{3}} \arctan\left(\frac{z}{\sqrt{3}}\right) = -\ln(u) + c$$

Now we must "backsubstitute" to write the solution in terms of the original variables. First $z = \frac{v}{u}$, so we get

$$\frac{1}{2}\ln((v/u)^2+3) + \frac{1}{\sqrt{3}} \arctan\left(\frac{v}{u\sqrt{3}}\right) = -\ln(u) + c.$$

Then from the original "homogenizing substitution", $t = u+1$ and $x = v-3$, so $u = t-1$ and $v = x+3$. We get

$$\frac{1}{2}\ln\left(\left(\frac{x+3}{t-1}\right)^2 + 3\right) + \frac{1}{\sqrt{3}} \arctan\left(\frac{x+3}{(t-1)\sqrt{3}}\right) = -\ln(t-1) + c.$$

The answer can be left in this form, or we can combine the two logarithm terms to obtain the answer in the text. Note:

$$\frac{1}{2}\ln\left(\left(\frac{x+3}{t-1}\right)^2 + 3\right) + \ln(t-1)$$

can be rewritten as

$$\frac{1}{2}\ln\left[\left(\left(\frac{x+3}{t-1}\right)^2 + 3\right)(t-1)^2\right]$$

or

$$\frac{1}{2}\ln((x+3)^2 + 3(t-1)^2)$$

since $\frac{1}{2}\ln(A) + \ln(B) = \frac{1}{2}\ln(AB^2)$.

§2.6/46. a) We assume that $u$ is a particular solution of the Riccati equation

$$\frac{dy}{dx} = P(x)y^2 + Q(x)y + R(x), \tag{5}$$

So

$$\frac{du}{dx} = P(x)u^2 + Q(x)u + R(x). \tag{6}$$

Then to find other solutions, we let $y = u + \frac{1}{v}$ as suggested in the problem and substitute into (5). We obtain

$$\begin{aligned}\frac{du}{dx} - \frac{1}{v^2}\frac{dv}{dx} &= P(x)\left(u + \frac{1}{v}\right)^2 + Q(x)\left(u + \frac{1}{v}\right) + R(x)\\ &= P(x)u^2 + 2P(x)\frac{u}{v} + \frac{P(x)}{v^2} + Q(x)u + \frac{Q(x)}{v} + R(x)\end{aligned}$$

Using (6), we can cancel terms from both sides to obtain:

$$-\frac{1}{v^2}\frac{dv}{dx} = 2P(x)\frac{u}{v} + \frac{P(x)}{v^2} + \frac{Q(x)}{v}$$

or after clearing denominators: $\frac{dv}{dx} = -2P(x)uv - P(x) - Q(x)v$, so

$$\frac{dv}{dx} + (2P(x)u + Q(x))v = -P(x)$$

which is first order linear.

b) To solve:

$$\frac{dy}{dx} = x^3(y-x)^2 + \frac{y}{x} = x^3y^2 + \left(-2x^4 + \frac{1}{x}\right)y + x^5$$

This is of the Riccati form (5) with $P(x) = x^3, Q(x) = -2x^4 + \frac{1}{x}, R(x) = x^5$ By the general derivation from part a), using the particular solution $u = x$ given in the problem, we see that $y = x + \frac{1}{v}$ solves the equation when $v$ solves

$$\frac{dv}{dx} + (2P(x)u + Q(x))v = -P(x)$$

or

$$\frac{dv}{dx} + \left(2x^3 \cdot x + -2x^4 + \frac{1}{x}\right)v = -x^3$$

or

$$\frac{dv}{dx} + \frac{1}{x}v = -x^3.$$

The integrating factor is $e^{\int \frac{1}{x}dx} = |x|$, and

$$v = \frac{c}{|x|} + \frac{1}{|x|}\int |x| \cdot (-x^3)dx$$

Assuming we want a solution for $x > 0$, this can be written as

$$= \frac{c}{x} - \frac{x^4}{5}$$

And then

$$\begin{aligned}y = u + \frac{1}{v} &= x + \frac{1}{\frac{c}{x} - \frac{x^4}{5}}\\ &= x + \frac{5x}{c - x^5}.\end{aligned}$$