Mathematics 304 – Ordinary Differential Equations
Problem Set 5 – Selected Solutions
November 13, 2000

Section 4.8

56. a) We want to find the general solution of the ODE:

$$mx'' + bx' + kx = \sin(\beta t) \tag{1}$$

The homogeneous equation $mx'' + bx' + kx = 0$ has auxiliary equation $mr^2 + br + k = 0$, so

$$r = \frac{-b \pm \sqrt{b^2 - 4mk}}{2m}.$$

Since we are given $b^2 < 4mk$, the two values for r are complex conjugates, and we can write the general solution of the homogeneous equation as

$$c_1 e^{\frac{-bt}{2m}} \cos(\omega t) + c_2 e^{\frac{-bt}{2m}} \sin(\omega t)$$

where $\omega = \frac{\sqrt{4mk - b^2}}{2m}$.

To solve the inhomogeneous equation, we use the undetermined coefficients method. By form III in the table on page 208 of the text, we try a particular solution of the form

$$x_p = A\cos(\beta t) + B\sin(\beta t)$$

(Note that neither term here can solve the homogeneous equation – no exponential terms – hence we take $s = 0$ in the undetermined coefficients formula.) Substituting x_p into (1), we get:

$$(k - m\beta^2)A + b\beta B = 0$$
$$-b\beta A + (k - m\beta^2)B = 1$$

The simplest way to write down the solution of a system like this is to use Cramer's Rule:

$$A = \frac{\det\begin{pmatrix} 0 & b\beta \\ 1 & k - m\beta^2 \end{pmatrix}}{\det\begin{pmatrix} k - m\beta^2 & b\beta \\ -b\beta & k - m\beta^2 \end{pmatrix}} = \frac{-b\beta}{(k - m\beta^2)^2 + (b\beta)^2}$$

and

$$B = \frac{\det\begin{pmatrix} k - m\beta^2 & 0 \\ -b\beta & 1 \end{pmatrix}}{\det\begin{pmatrix} k - m\beta^2 & b\beta \\ -b\beta & k - m\beta^2 \end{pmatrix}} = \frac{k - m\beta^2}{(k - m\beta^2)^2 + (b\beta)^2}$$

Then the general solution of the equation (1) is

$$c_1 e^{\frac{-bt}{2m}} \cos(\omega t) + c_2 e^{\frac{-bt}{2m}} \sin(\omega t) + A\cos(\beta t) + B\sin(\beta t),$$

where A, B are given by the formulas above.

b) Because $b/2m > 0$, the exponentials in the general solution tend to zero as $t \to \infty$. The solution tends to $A\cos(\beta t) + B\sin(\beta t)$. This is a sinusoidal function of the form $C = \cos(\beta t + \phi)$ for suitable C, ϕ.

57. The idea here is to follow the same method used in problem 55. We first solve the IVP:

$$\begin{cases} y'' + y = \cos(Vt) \\ y(-\pi/(2V)) = 0 \\ y'(-\pi/(2V)) = 0 \end{cases} \tag{2}$$

(Note: it is given that $y(t) = 0$ for all $t \leq -\pi/(2V)$ – the car is not moving vertically at all before it hits the speed bump. So this gives the initial conditions at $t = -\pi/(2V)$.) Call the solution of (2) $y_a(t)$. *Then*, at $t = \pi/(2V)$, we solve a second IVP:

$$\begin{cases} y'' + y = 0 \\ y(\pi/(2V)) = y_a(\pi/(2V)) \\ y'(\pi/(2V)) = y_a'(\pi/(2V)) \end{cases} \tag{3}$$

to get the continuation of the solution to $t > \pi/(2V)$.

The first step is to find the function $y_a(t)$ solving (2). The general solution of the homogeneous equation $y'' + y' = 0$ is $y = c_1\cos(t) + c_2\sin(t)$. By the form of the right hand side of the ODE in (2), for $V \neq 1$, we expect a particular solution of the form

$$y_p = A\cos(Vt) + B\sin(Vt).$$

Substituting into (2), we find that

$$A(1 - V^2)\cos(Vt) + B(1 - V^2)\sin(Vt) = \cos(Vt)$$

so $A = \frac{1}{1-V^2}$ and $B = 0$.

Now, we determine the constants c_1, c_2 in the general solution

$$y = c_1\cos(t) + c_2\sin(t) + \frac{1}{1 - V^2}\cos(Vt)$$

to get $y(-\pi/(2V)) = 0$ and $y'(-\pi/(2V)) = 0$. These initial conditions say:

$$\begin{aligned} c_1\cos(-\pi/(2V)) + c_2\sin(-\pi/(2V)) + 0 &= 0 \\ -c_1\sin(-\pi/(2V)) + c_2\cos(-\pi/(2V)) &= \frac{V}{V^2 - 1} \end{aligned}$$

By Cramer's Rule again,

$$c_1 = \frac{\det\begin{pmatrix} 0 & \sin(-\pi/(2V)) \\ \frac{V}{V^2-1} & \cos(-\pi/(2V)) \end{pmatrix}}{\det\begin{pmatrix} \cos(-\pi/(2V)) & \sin(-\pi/(2V)) \\ -\sin(-\pi/(2V)) & \cos(-\pi/(2V)) \end{pmatrix}} = \frac{V\sin(\pi/(2V))}{V^2-1}$$

$$c_2 = \frac{\det\begin{pmatrix} \cos(-\pi/(2V)) & 0 \\ -\sin(-\pi/(2V)) & \frac{V}{V^2-1} \end{pmatrix}}{\det\begin{pmatrix} \cos(-\pi/(2V)) & \sin(-\pi/(2V)) \\ -\sin(-\pi/(2V)) & \cos(-\pi/(2V)) \end{pmatrix}} = \frac{V\cos(\pi/(2V))}{V^2-1}$$

This gives the coefficients in $y_a(t)$, the formula for the solution on $-\pi/(2V) < t < \pi/(2V)$.

Now, we want to solve the second IVP (3). If we call the solution $y_b(t)$, then from the ODE in (3) – the homogeneous equation $y''+y=0$, we know that $y_b = d_1\cos(t)+d_2\sin(t)$, and we must determine d_1, d_2 so that

$$y_b(\pi/(2V)) = y_a(\pi/(2V)) = c_1\cos(\pi/(2V))+c_2\sin(\pi/(2V)) = \frac{2V\sin(\pi/(2V))\cos(\pi/(2V))}{V^2-1}$$

and

$$y_b'(\pi/(2V)) = y_a'(\pi/(2V)) = \frac{2V\cos^2(\pi/(2V))}{V^2-1}.$$

So,

$$\begin{aligned} \cos(\pi/(2V))d_1 + \sin(\pi/(2V))d_2 &= \frac{2V\sin(\pi/(2V))\cos(\pi/(2V))}{V^2-1} \\ -\sin(\pi/(2V))d_1 + \cos(\pi/(2V))d_2 &= \frac{2V\cos^2(\pi/(2V))}{V^2-1} \end{aligned}$$

Solving by Cramer's Rule, we find $d_1 = 0$, and $d_2 = \frac{2V\cos(\pi/(2V))}{V^2-1}$. Thus the solution on $t \geq \pi/(2V)$ is

$$y_b(t) = \frac{2V\cos(\pi/(2V))}{V^2-1}\sin(t)$$

Plotting the amplitude as a function of V, we find $A(V) = \frac{2V\cos(\pi/(2V))}{V^2-1}$ has a maximum at roughly $V \doteq .73$.