MATH 304 – Ordinary Differential Equations
Final Exam Solutions

I.

A) By the Chain Rule, and since $y = \phi(x)$ solves $\frac{dy}{dx} = f(y)$,

$$\psi'(x) = \phi'(x - x_0) \cdot 1 = f(\phi(x - x_0)) = f(\psi(x))$$

In addition, $\psi(x_0) = \phi(x - x_0) = \phi(0) = \alpha$.

B) The equilibria are the roots of $f(x) = 0$, at $x = -2, 3/2$ (approximately), and 4.

C) The solution of the IVP with $\alpha = 1$ is always decreasing since $f(y) < 0$ for all y with $-2 < y < 3/2$. It satisfies $\lim_{x\to\infty} \phi(x) = -2$.

D) The solution of the IVP with $\alpha = 2$ is always increasing since $f(y) > 0$ for all y with $3/2 < y < 4$. It satisfies $\lim_{x\to\infty} \phi(x) = 4$.

II.

A) After dividing by t^3, the equation has the form

$$\frac{dz}{dt} + \frac{-4}{t} z = t^5 \sin(t)$$

which is *first order linear*. The integrating factor is $e^{\int -4/t\ dt} = t^{-4}$, so

$$z = ct^4 + t^4 \int \frac{1}{t^4} t^5 \sin(t)\ dt = ct^4 + t^4 \int t \sin(t)\ dt$$

Integrating by parts, we have

$$z = ct^4 + t^4(\sin(t) - t\cos(t))$$

B) This is a *homogeneous* equation. Letting $v = \theta/s$, we get $\frac{d\theta}{ds} = s\frac{dv}{ds} + v$, so

$$s\frac{dv}{ds} + v = \frac{4 - 3v}{2 - v}$$

After some algebra:

$$\frac{dv}{ds} = \frac{4 - 5v + v^2}{2 - v} = \frac{(v-1)(v-4)}{2 - v}$$

which is separable:

$$\int \frac{2 - v}{(v-1)(v-4)}\ dv = \int \frac{1}{s} ds$$

To integrate the left side, use the entries from the table of integrals in the textbook for rational functions with a factorable quadratic downstairs ($b = \beta = 1, a = -4, \alpha = -1$):

$$\frac{-2}{3} \ln\left|\frac{v-1}{v-4}\right| + \frac{1}{3}(-4\ln|v-4| + \ln|v-1|) = \ln|s| + c$$

This can be simplified to yield

$$-2\ln|v-4| - \ln|v-1| = 3\ln|s| + c$$

so after exponentiating

$$(v-4)^2(v-1) = c/s^3$$

Substituting $v = \theta/s$ yields the final answer.

C) Rearranging to

$$(\cos(y) - e^x)dx + (-y^2 - x\sin(y))dy = 0$$

the equation is seen to be *exact*. Solutions

$$x\cos(y) - e^x - y^3/3 = c$$

III.

A) The time constant is $1/k = 3/2$, hence $k = 2/3$. Putting $t = 0$ at 1 am, the outside temperature can be written as $M(t) = 55 - 15\cos(\pi t/12)$. Hence by Newton's Law of Heating and Cooling, the differential equation is:

$$\frac{dT}{dt} = \frac{2}{3}(55 - 15\cos(\pi t/12) - T)$$

Rearranging, we have

$$\frac{dT}{dt} + \frac{2}{3}T = \frac{2}{3}(55 - 15\cos(\pi t/12)),$$

which is *first order linear*. Using the entry from the table of integrals for $\int e^{au}\cos(nu)\,du$, the solution is:

$$\begin{aligned}
T &= ce^{-2t/3} + e^{-2t/3}\int e^{2t/3}\frac{2}{3}(55 - 15\cos(\pi t/12))\,dt \\
&= ce^{-2t/3} + 55 - 10e^{-2t/3}\int e^{2t/3}\cos(\pi t/12))\,dt \\
&= ce^{-2t/3} + 55 - \frac{10}{\frac{4}{9} + \frac{\pi^2}{144}}(\frac{2}{3}\cos(\pi t/12) + \frac{\pi}{12}\sin(\pi t/12)) \\
&\doteq ce^{-2t/3} + 55 - 12.99\cos(\pi t/12) - 5.10\sin(\pi t/12)
\end{aligned}$$

B) After the exponential term dies off, the solution is approximately

$$T = 55 - 12.99\cos(\pi t/12) - 5.10\sin(\pi t/12)$$

Using the identity $A\cos(\omega t) + B\sin(\omega t) = C\cos(\omega t - \phi)$ where $C = \sqrt{A^2 + B^2}$ and $\phi = \arctan(B/A)$, we see that the maximum temperature is attained when $\pi t/12 - \arctan(5.10/12.99) = \pi$, or

$$t \doteq \frac{12}{\pi}(\pi + .37) \doteq 13.41$$

This means approximately 13.41 hours after 1am, or 2:24 pm (and every 24 hours thereafter).

IV. We have $y_2 - y_1 = 3x$ and $y_1 - y_3 = 3x^2 - x$ are solutions of the homogeneous equation $y'' + py' + q = 0$. They are linearly independent because they are polynomials of two different degrees. Hence by Abel's Identity,

$$Ce^{-\int p(x)\ dx} = W[3x, 3x^2 - x] = 9x^2$$

for some constant C. Taking logarithms and differentiating, we find

$$p(x) = \frac{-2}{x}$$

V.

A) $y = c_1 e^{-4x} + c_2 x e^{-4x}$

B) The particular solution should have the form $Ax^2e^{-4x} + Bx^2 + Cx + D$. Solving for A, B, C, D yields

$$y_p = \frac{3}{2}x^2e^{-4x} + \frac{1}{16}x^2 + \frac{-5}{16}x + \frac{35}{128}$$

C) If the 16 is replaced by $16 + \epsilon$, $\epsilon > 0$, then the roots of the auxiliary equation become

$$r = -4 \pm i\sqrt{\epsilon}$$

The form of the homogeneous solutions changes to

$$y = c_1 e^{-4x}\cos(\sqrt{\epsilon}x) + c_2 e^{-4x}\sin(\sqrt{\epsilon}x)$$

and the form of the particular solution changes to $Ae^{-4x} + Bx^2 + Cx + D$ since e^{-4x} is no longer a solution of the homogeneous equation.

VI.

A) The system has two critical points $(0, 4)$ and $(-6, -2)$.

B) The linearization at $(0, 4)$ has the matrix

$$\begin{pmatrix} -1 & 1 \\ -12 & 0 \end{pmatrix}$$

with eigenvalues $\lambda = \frac{-1\pm\sqrt{-47}}{2}$ (complex conjugates with negative real parts). This is an *inward spiral* point. The linearization at $(-6, -2)$ has the matrix

$$\begin{pmatrix} -1 & 1 \\ -6 & 18 \end{pmatrix}$$

with eigenvalues $\lambda = \frac{17\pm\sqrt{337}}{2}$ (real with opposite signs). This is a *saddle* point.

C) Solutions of linearized system at $(-6, -2)$: Using the decimal approximations $\lambda \doteq -.678, 17.7$, we get corresponding eigenvectors:

$$\begin{pmatrix} 1 \\ .322 \end{pmatrix} \qquad \begin{pmatrix} 1 \\ 18.7 \end{pmatrix}$$

so the solutions are

$$c_1 e^{-.678t}\begin{pmatrix} 1 \\ .322 \end{pmatrix} + c_2 e^{17.7t}\begin{pmatrix} 1 \\ 18.7 \end{pmatrix}$$