Mathematics 304 – Ordinary Differential Equations
Lab 3 – RLC Circuits and 2nd Order Equations
October 26, 2000

*Background and Goals*

Today we will apply our knowledge of 2nd order linear ODE to some questions about electrical circuits. An "RLC circuit" is one that contains a voltage source $v_T(t)$ (time dependent), a *resistor* (i.e. something that *dissipates electrical energy*) with resistance $R$, an *inductor* (a device or component that *stores magnetic energy*) with inductance $L$, and a *capacitor* (a device or component that *stores electrical energy*) with capacitance $C$. Many basic electrical circuits have this form.

An RLC circuit

From physics, it can be derived that the voltage change across the capacitor, written $v_C$, satisfies the 2nd order ODE

$$LC\frac{d^2v_C}{dt^2} + RC\frac{dv_C}{dt} + v_C = v_T(t) \tag{1}$$

We will always consider a *periodic, sinusoidal* voltage source ("forcing") term

$$v_T(t) = a\sin(\omega t).$$

*Lab Questions*

A) (A worked example – you supply the details, graph, etc.) For instance, suppose we want to understand the voltage in an RLC circuit with $R = 1000$, $L = 2$, and $C = 10^{-6}$, and a voltage source $v_T = 20\sin(500t)$. From class on Tuesday, we know that the general solution of (1) in this case:

$$(2\times 10^{-6})\frac{d^2v_C}{dt^2} + (10^{-3})\frac{dv_C}{dt} + v_C = 20\sin(50t) \tag{2}$$

looks like $v_C = c_1 v_1 + c_2 v_2 + v_p$, where $\{v_1, v_2\}$ is a fundamental set of solutions for the homogeneous equation

$$(2 \times 10^{-6})\frac{d^2 v_C}{dt^2} + (10^{-3})\frac{dv_C}{dt} + v_C = 0$$

and $v_p$ is some particular solution.

1) To find $v_1, v_2$, we find the roots of the auxiliary equation of the ODE. What are they here? What does the corresponding general solution of the homogeneous equation look like?
2) Now, we need to find a particular solution $v_p$. This is where the method of undetermined coefficients that we discussed in class on Tuesday comes in. The $20\sin(50t)$ term on the right is a very simple special case of the third general case we discussed in class. (It's also a special case of class V in the table on page 208 of the text.) We guess that a particular solution will have the form $v_p = A\cos(50t) + B\sin(50t)$. (Note, no term here is a solution of the homogeneous equation, so we *do not need the $t^s$ factor*. Take $s = 0$.) Plug $v_p$ into (2) and determine the values of $A, B$ to yield a solution. You should get:

$$.995A + .05B = 0$$
$$-.05A + .995B = 20$$

   So $A \doteq -1.007$ and $B \doteq 20.05$.
3) Plot the solution of (2) with $v_C(0) = 0$, and $v_C'(0) = 1$. (Note: You'll need to think about an appropriate interval of $t$ values for your plot. A term like $\cos(661.437t)$ is oscillating really fast!)

B) Suppose $L, C > 0$, but the resistance $R = 0$. (This is a very, very idealized situation, of course, even better than "superconductivity.") What do the solutions of (1) look like in this case? Derive analytic formulas by our *undetermined coefficients* technique for inhomogeneous 2nd order equations. Be sure to distinguish between the two different cases:
   1) $i\omega$ is not a root of the auxiliary equation, and
   2) $i\omega$ is a root of the auxiliary equation.

   How is the analytic form of the solution different in these two cases?

C) Using Maple, plot the analytic solutions of (1) with $R = 0$, $C = 10^{-6}$, $L = 1$, $a = 1$, $v_C(0) = 1$, $v_C'(0) = 0$, and $\omega = 900, 950, 1000, 1050, 1100$ (five graphs in all). (You'll need to chose your $t$ interval carefully to get informative graphs!) The name for the $\omega = 1000$ case is *resonance*. Describe what a resonant solution does as $t \to \infty$.

D) Next, we will study the more realistic case $R > 0$. By considering the form of the roots of the auxiliary equation, show that if $R > 0$, every solution of the corresponding homogeneous equation

$$LC\frac{d^2 v_C}{dt^2} + RC\frac{dv_C}{dt} + v_C = 0$$

tends to *zero* as $t \to \infty$. These terms in the general solution of (1) are called "transients" for this reason.

E) Plot and describe the solutions for various values of $a$ and $\omega$ when $R = 2000$, $C = 2 \times 10^{-7}$, and $L = 1.5$.

F) What types of long term behavior are possible for solutions of (1) when $R > 0$? Does resonance occur in this situation?

*Maple Notes*

All of the plots necessary for this lab will be generated using the basic Maple `plot` command. However, if you want to check your results, you can also use the `DEplot` command from Lab 1 to plot approximate numerical solutions to 2nd order ODE as follows. For example, say we want to see the solution of the initial value problem

$$\begin{cases} 4\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 10y = 3\cos(2t) \\ y(0) = 5 \\ y'(0) = -3 \end{cases}$$

Then the following commands could be used. First load the `DEtools` package with:

```
with(DEtools):
```

Then define the ODE with:

```
eqn:=4*diff(y(t),t$2)+3*diff(y(t),t)+10*y(t)  = 3*cos(2*t);
```

and plot the solution for $t \in [0, 10]$ with

```
DEplot(eqn,y(t),t=0..10,[[y(0)=5,D(y)(0)=-3]]);
```

As always, adjusting the `linecolor` and `stepsize` options may be necessary to get reasonable, printable graphs.

*Assignment*

Individual write-ups giving answers to the above questions. **Due:** In class, November 2.