Mathematics 304 – Ordinary Differential Equations
Discussion/Lab 2 – Modelling with ODE's
Competition for a Heating/Air Conditioning Plan for the new Smith Hall
September 21 and 26, 2000

Background and Goals

Last time, we saw that if we think of the air volume in a building as one "compartment" at uniform temperature and we want to model how the temperature changes over time, then we need to analyze the different factors affecting the temperature such as:

- heat exchange with the surrounding air
- heat sources (such as computers, lights, other machinery, etc.) within the building
- air conditioning/heating applied to control the temperature.

We will write $T(t)$ for the air temperature in the building as a function of time. By *Newton's Law of Heating and Cooling*, the rate of change of T with respect to time *due to heat exchange with the surrounding air* is proportional to the difference between the the temperature of the building and the ambient air temperature $A(t)$ (which will depend on t). If heat sources within the building are changing the temperature by $H(t) \geq 0$ at time t and heating or air conditioning are changing the temperature by $U(t,T)$ at time t, our model for the real world system is a 1st order ODE:

$$\frac{dT}{dt} = k(T - A(t)) + H(t) + U(t,T), \tag{1}$$

where k is constant, and $A(t), H(t)$ are functions of t alone (assumed known). We will think of the $U(t,T)$ term as the *control term* that we can adjust to try to produce solutions with desirable properties (in particular, we will want T to be as nearly *constant* as possible.)

With your knowledge of ODE's, your discussion group has been hired by Perini Construction to help design an air conditioning/heating plan for the new Smith Hall under construction at Holy Cross College. The goal is to develop a plan to keep the building as close as possible to a constant temperature, given information about the ambient temperature $A(t)$ and the building's heat exchange characteristics. Unbeknownst to you, Perini has also hired all the other discussion groups in the class and given them the same problem (sneaky!!) They will use the best solution for the building in question, and the "winner" will receive a nice bonus. So, it's in your best interest to get as good a solution as possible!

Discussion/Lab Questions

A) If we assume that $U(t,T)$ depends "at most linearly" on T, that is: $U(t,T) = U_0(t) + U_1(t)T$, with $U_1(t)$ possibly identically zero, what kind of differential equation is (1)? Show how to derive a general solution in that case.

B) By considering what happens with solutions of (1) in the case $H(t) \equiv 0$ and $U(t, T) \equiv 0$ and $A(t)$ constant, what sign should the constant k have? What does the magnitude of k tell you? What could you do in an actual building to change the value of k?

C) The building your group will work with has, of course, not yet been completed. But based on the building materials to be used. placement of windows and doors, etc., estimates have been developed by Perini to show that over a period of hours on a summer day with constant ambient temperature $A = 75°$F, with everything (i.e. all machinery in the building) *turned off*, if the building air temperature was 68°F at some time, then the temperature 2 hours later would be was 72°F. What useful information does this give you?

D) In order to design an air conditioning/heating plan over an extended period (e.g. a full year), we need to make some assumptions about what the ambient temperature will look like as a function of time. To derive our model, we will assume that the temperature varies periodically with period one year. Since Smith Hall is located in Worcester, MA, a *"typical" year* will have minimum average temperature 25°F on February 1. The average temperature each day will vary continuously up to a a maximum average temperature of 80°F on August 1, then decrease back down to 25°F by the next February 1. Over each day, the temperature will vary from a minimum of 8 degrees below the average for that day to a maximum of 8 degrees above. Develop a possible formula giving the temperature function $A(t)$ as a function of t satisfying this description. You can choose any convenient unit of time. (Note: more realistic ways to model this might be to use actual temperature data from several "representative years" or to include some randomness in the ambient temperature function. Either way would be more involved, so we'll stick to the exactly periodic $A(t)$!)

E) We'll assume that the effects of heat sources within Smith Hall are constant and relatively small in size. Take $H(t) \equiv 0.1$ (degrees F per hour). Using your $A(t)$ from question D and the appropriate k value, plot the solution of (1) with no heating/air conditioning control applied (that is with $U(t, T) \equiv 0$) using Maple's DEplot command and/or by plotting an analytic solution of (1). (See the instructions for Lab 1 to refresh your memory on this!) Do we really need heating/air conditioning?? Explain.

F) Two standard types of control functions $U(t, T)$ that try to make T agree with a specified *desired temperature* T_D are:

- "Bang-bang" controls, where $U(t, T)$ is *piecewise constant*. Think of cranking up the air conditioner to a constant level when the inside temperature is higher than the desired temperature (for instance in the summer months), so that contribution of this term to the rate of change of T is a negative constant for some ranges of t-values, and cranking up the furnace to a constant level when the inside temperature is lower than the desired temperature (through the winter months), so that contribution of this term to the rate of change of T is a positive constant for some other ranges of

t-values. For this method, you get to choose the constant rates of heating and cooling that apply while $T \neq T_D$. (But see question H below!)

- "Proportional" controls, where $U(t, T)$ is proportional to the difference between the actual temperature T and the desired temperature T_D. In this case, you get to choose the constant of proportionality. (But see question H below!)

Both these types of control can be implemented easily with standard hardware, so you have the choice of using either one. (For instance, basic thermostats like the ones in most houses are a variation of the first of these two ideas: they turn on when the temperature gets outside a specified range around a desired temperature, and apply a constant rate of heating or cooling while they are on.) Experiment with both types of controls and plot the corresponding solutions in Maple to try to get the temperature for Smith Hall as close as you can to a comfortable constant $T_D \equiv 70°\text{F}$ over the course of a year. Explain how you are measuring the deviation from the constant desired temperature. Which method seems to be superior? Explain your assertion.

Maple Notes

To implement the "bang-bang" idea in Maple, you will probably want to use a procedure to compute values for the control function $U(t, T)$. There is some info about programming in Maple on the course homepage – look at the Maple programming primer. Also, if you do things that way, there is apparently not a simple way to get Maple to plot approximate solutions using the DEplot command. You will want to use the Maple code for the Runge-Kutta method that is available from the course homepage. This will also be demonstrated in the lab class on September 26.

G) Describe your "best-shot" heating/cooling plan for Smith Hall (your most successful function $U(t, T)$ – the one that comes closest to the constant temperature goal according to your specified measure).

H) Suppose that applying heating or cooling control of ± 1 degree per hour for a time period of one hour costs \$30 (probably an underestimate for a building the size of Smith Hall). Explain how to estimate the total heating costs for one year. If you have several possible heating plans that seem equally good or bad, is there a choice between them suggested by this consideration? What would be the cost of your best plan?

I) What real-world considerations does your model *ignore* that might be important for an HVAC engineer to take into account?

Assignment

Group write-ups giving answers to the above questions, and presenting a proposed heating plan for Smith Hall giving the type of controls – bang-bang or proportional – you want to use, a plot of the corresponding temperature function over the year according to your plan, and an estimate of the cost to apply your plan. **Due:** In class, October 3. May the best team win!