Mathematics 304 – Ordinary Differential Equations
Discussion/Lab 3 – RLC Circuits and 2nd Order Equations
October 26, 2000

Background and Goals

Today we will apply our knowledge of 2nd order linear ODE to some questions about electrical circuits. An RLC circuit is one that contains a voltage source $v_T(t)$ (time dependent), a *resistor* (i.e. something that *dissipates electrical energy*) with resistance R, an *inductor* (a device or component that *stores magnetic energy*) with inductance L, and a *capacitor* (a device or component that *stores electrical energy*) with capacitance C.

An RLC circuit

From physics, it can be derived that the voltage change across the capacitor, written v_C, satisfies the 2nd order ODE

$$(1). \qquad LC\frac{d^2 v_C}{dt^2} + RC\frac{dv_C}{dt} + v_C = v_T(t)$$

We will always consider a *periodic, sinusoidal* voltage source ("forcing") term

$$v_T(t) = a\sin(\omega t).$$

Discussion/Lab Questions

A) Suppose $L, C > 0$, but $R = 0$. (This is a very idealized situation, of course, even better than "superconductivity.") What do the solutions of (1) look like in this case. Derive analytic formulas by our techniques for inhomogeneous 2nd order equations. Be sure to distinguish between the two different cases:
   1) $i\omega$ is not a root of the auxiliary equation, and
   2) $i\omega$ is a root of the auxiliary equation.
   How is the analytic form of the solution different in these two cases?

B) Using Maple, plot the analytic solutions of (1) with $R = 0$, $C = 10^{-6}$, $L = 1$, $a = 1$, $v_C(0) = 1$, $v_C'(0) = 0$, and $\omega = 900, 950, 1000, 1050, 1100$ (five graphs in all). (You'll need to chose your t interval carefully to get informative graphs!) The name for the $\omega = 1000$ case is *resonance*. Describe what a resonant solution does as $t \to \infty$.

C) Next, we will study the more realistic case $R > 0$. By considering the form of the roots of the auxiliary equation, show that if $R > 0$, every solution of the corresponding homogeneous equation

$$LC\frac{d^2 v_C}{dt^2} + RC\frac{dv_C}{dt} + v_C = 0$$

tends to *zero* as $t \to \infty$. These terms in the general solution of (1) are called "transients" for this reason.

D) Plot and describe the solutions for various values of a and ω when $R = 2000$, $C = 2 \times 10^{-7}$, and $L = 1.5$.

E) What types of long term behavior are possible for solutions of (1) when $R > 0$? Does resonance occur in this situation?

Maple Notes

All of the plots necessary for this lab will be generated using the basic Maple `plot` command. However, if you want to check your results, you can also use the `DEplot` command from Lab 1 to plot approximate numerical solutions to 2nd order ODE as follows. For example, say we want to see the solution of the initial value problem

$$\begin{cases} 4\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 10y = 3\cos(2t) \\ y(0) = 5 \\ y'(0) = -3 \end{cases}$$

Then the following commands could be used. First load the `DEtools` package with:

```
with(DEtools):
```

Then define the ODE with:

```
eqn:=4*diff(y(t),t$2)+3*diff(y(t),t)+10*y(t)  = 3*cos(2*t);
```

and plot the solution for $t \in [0, 10]$ with

```
DEplot(eqn,y(t),t=0..10,[[y(0)=5,D(y)(0)=-3]]);
```

As always, adjusting the `linecolor` and `stepsize` options may be necessary to get reasonable, printable graphs.

Assignment

Group write-ups giving answers to the above questions. **Due:** In class, November 2.