Mathematics 304 – Ordinary Differential Equations
Discussion 1 – Qualitative Techniques for 1st Order ODE
August 31, 2000

Background

For a first order ODE $\frac{dy}{dx} = f(x, y)$, the *direction field* gives valuable qualitative information about solutions. The direction field is a plot in the (x, y)-plane, where a small line segment of slope $f(x, y)$ is placed at the point (x, y). Conceptually, we do this at all points where $f(x, y)$ is defined, but that is impossible to draw, so some finite collection of "representative" points (x, y) is used in practice.

To draw slope fields by hand, a useful technique is:

1. First find the *isoclines* of the equation – the curves of constant slope value given by the equations $f(x, y) = c$. (These are the same as the *level curves* of the slope function $f(x, y)$ as defined in multivariable calculus.)
2. Sketch some collection of the isoclines (often the isoclines for integer slope values, e.g.

$$c = \ldots, -2, -1, 0, 1, 2, \ldots$$

   are sufficient to see what is happening)
3. Draw in line segments of slope c at points on the isocline for the slope value c.

Then, for instance, we can often get qualitative information about the solutions, since a solution $y = \phi(x)$ is a function whose graph "follows the direction field" – at each point on the graph, the slope of the tangent line satisfies

$$\phi'(x) = \frac{dy}{dx} = f(x, \phi(x)).$$

From this picture, the following should be plausible:

Existence and Uniqueness Theorem for ODE Initial Value Problems. *Let $f(x, y)$ and $\frac{\partial f(x,y)}{\partial y}$ be defined and continuous on a rectangle R in the plane containing (x_0, y_0). Then there is a unique solution $\phi(x)$ of the IVP*

$$\begin{cases} \frac{dy}{dx} = f(x, y) \\ y(x_0) = y_0 \end{cases}$$

defined on some open interval containing x_0.

Today, in our first discussion, we will study direction fields for some differential equations and see what they tell us about solutions.

Discussion Questions

A) Sketch the isoclines with $c = -2, -1, 0, 1, 2$ for the differential equation

$$\frac{dy}{dx} = y - x$$

and use them to sketch the direction field. What can you say about the function $\phi(x) = x + 1$ (is it a solution of this equation?) Give rough qualitative sketches of the solutions $\phi(x)$ with $\phi(0) = 0$ and $\phi(-2) = 1$. Describe the graphs of solutions of the initial value problems:

$$\begin{cases} \frac{dy}{dx} = y - x \\ y(-1) = \alpha \end{cases}$$

What happens as $x \to +\infty$? How does your answer depend on the value of α?

B) A first order ODE whose slope function is *independent of* x (that is, which is of the form $\frac{dy}{dx} = f(y)$) is called an *autonomous* equation.

1) What do the isoclines of an autonomous equation look like? (What kind(s) of curve(s) are they?)
2) Suppose that f, f' are continuous on all of $\mathbf{R}$. If x_0 is an arbitrary x-value, show that the Existence and Uniqueness Theorem for 1st order ODE implies that there is one and only one solution of the IVP

$$\begin{cases} \frac{dy}{dx} = f(y) \\ y(x_0) = y_0 \end{cases}$$

Explain why this says that the graphs of two different solutions of an autonomous equation *never intersect*.
3) Suppose that $f(y_0) = 0$ for some particular y_0. Explain why these are called *equilibrium points* for the ODE. What can you say about the constant function $\phi(x) = y_0$? (Is it a solution of the equation?)
4) For each of the following autonomous ODE, find all the equilibrium points, sketch the part of the direction field for $-4 \le y \le 4$ and a "representative sampling" of solutions:

$$\frac{dy}{dx} = -y^2 + 5y - 6 = (y-2)(3-y)$$
$$\frac{dy}{dx} = y^3 - 2y^2 + y = y(y-1)^2$$
$$\frac{dy}{dx} = y\cos(2y)$$

5) One important qualitative question about solutions of ODE is: *what is their long-term behavior*? That is, what is $\lim_{x\to\infty} \phi(x)$? In this part, you will study how the graph of the slope function (i.e. the graph $w = f(y)$) can help determine this. In particular, when is it true that all solutions starting from points near $y = y_0$

tend toward $y = y_0$ as $x \to \infty$? When is it true that all solutions starting near an equilibrium move away? Is the behavior always the same on both sides of an equilibrium? Use your work from part 4 to help formulate a general statement about how the graph $w = f(y)$ relates to the behavior of solutions near equilibria, and justify your statements.

Assignment

Group write-ups due in class on Tuesday, September 5.