Mathematics 371 – Numerical Analysis
Selected Solutions – Problem Set 3
October 19, 2001

Section 3.1/3a and 7a. Given:

x_i	$f(x_i)$
8.1	16.94410
8.3	17.56492
8.6	18.50515
8.7	18.82091

To approximate $f(8.4)$ with an interpolating polynomial of degree 1, we would almost always choose the two closest points "bracketing" 8.4 – here 8.3 and 8.6. The Lagrange polynomial is

$$p_1(x) = 17.56492(x - 8.6)/(8.3 - 8.6) + 18.50515(x - 8.3)(8.6 - 8.3) = 3.1341x - 8.44811$$

and $p_1(8.4) = (3.1341)(8.4) - 8.44811 = 17.87833$.

With a polynomial of degree 2, we can choose any 3 x_i "bracketing" 8.4. The book's choice was 8.3,8.6,8.7:

$$\begin{aligned} p_2 = {} & 17.56492\frac{(x-8.6)(x-8.7)}{(8.3-8.6)(8.3-8.7)} + 18.50515\frac{(x-8.3)(x-8.7)}{(8.6-8.3)(8.6-8.7)} \\ & + 18.82091\frac{(x-8.3)(x-8.6)}{(8.7-8.3)(8.7-8.6)} \end{aligned}$$

Then $p_2(8.4) = 17.877155$.

Finally for the polynomial of degree 3, we use all 4 points

$$\begin{aligned} p_3 = {} & 16.94410\frac{(x-8.3)(x-8.6)(x-8.7)}{(8.1-8.3)(8.1-8.6)(8.1-8.7)} + 17.56492\frac{(x-8.1)(x-8.6)(x-8.7)}{(8.3-8.1)(8.3-8.6)(8.3-8.7)} \\ & + 18.50515\frac{(x-8.1)(x-8.3)(x-8.7)}{(8.6-8.1)(8.6-8.3)(8.6-8.7)} + 18.82091\frac{(x-8.1)(x-8.3)(x-8.6)}{(8.7-8.1)(8.7-8.3)(8.7-8.6)} \end{aligned}$$

And $p_3(8.4) = 17.877155$.

Since in problem 7a it is given that $f(x) = x\ln(x)$, we have actual errors:

$$\begin{aligned} |f(8.4) - p_1(8.4)| &= |18.87714633 - 17.87833| = .00118367 \\ |f(8.4) - p_2(8.4)| &= |18.87714633 - 17.877155| = .00000867 \end{aligned}$$

The error bounds for interpolation come from the error formula:

$$f(x) = p_n(x) + \frac{f^{(n+1)}(\xi)}{(n+1)!}(x - x_0)(x - x_1)\cdots(x - x_n)$$

For $n = 1$ here, $f''(\xi) = 1/\xi$, so $|f''(\xi)| \le 1/8.3$ on the interval $[8.3, 8.6]$ that we used to get p_1. We get:

$$|f(8.4) - p_1(8.4)| \le \frac{1}{2 \cdot 8.3}|8.4 - 8.3||8.4 - 8.6| \doteq .001205$$

which is just larger than the actual error. Similarly $f'''(x) = -1/x^2$, so

$$|f(8.4) - p_3(8.4)| \le \frac{1}{6 \cdot (8.3)^2}|8.4 - 8.3||8.4 - 8.6||8.4 - 8.7| \doteq .0000145$$

and the actual error is within the error bound again.

3.1/17. To construct the table as described, we want to make sure that the error bound for linear interpolation ($n = 1$) is less than 10^{-6} for all x with $1 \le x \le 10$, and $f(x) = \log(x)$ (common logarithm). We have: $f(x) = \ln(x)/\ln(10)$, so $f''(x) = \frac{-1}{x^2 \ln(10)}$. For $1 \le x \le 10$ this is largest at $x = 1$. Then

$$|\log(x) - p_1(x)| \le \frac{1}{2\ln(10)}|x - x_i| \cdot |x - (x_i + h)|$$

if we use the table entries for x_i and $x_{i+1} = x_i + h$. For x between x_i and $x_i + h$, the maximum of $|x - x_i| \cdot |x - (x_i + h)| = |(x - x_i)(x - (x_i + h)|$ occurs at $x = x_i + h/2$ and equals $(h/2)^2$. Hence our error bound gives:

$$|\log(x) - p_1(x)| \le \frac{1}{2\ln(10)}\left(\frac{h}{2}\right)^2 = \frac{h^2}{8\ln(10)}$$

Then

$$\frac{h^2}{8\ln(10)} < 10^{-6} \Leftrightarrow h < \sqrt{8\ln(10) \cdot 10^{-6}} \doteq .00429$$

Taking $h = .004$ (exactly 250 entries for between one integer x and the next will make $x = 10$ one of the entries.

3.1/29c. For this part, you need to use *generalizations* of the results from part a on the "Bernstein polynomials" for 1 and x:

$$\sum_{k=0}^{m} \binom{m}{k} \left(\frac{k}{m}\right) x^k (1 - x)^{m-k} = x \ (*)$$

$$\sum_{k=0}^{m} \binom{m}{k} x^k (1 - x)^{m-k} = 1 \ (**)$$

Then for $f(x) = x^2$, using part b and the fact that the term in the sum for $B_n(x)$ with $k = 0$ is equal to zero, we have

$$
\begin{aligned}
B_n(x) &= \sum_{k=0}^{n} \binom{n}{k} \left(\frac{k}{n}\right)^2 x^k (1-x)^{n-k} \\
&= \sum_{k=0}^{n} \binom{n-1}{k-1} \left(\frac{k}{n}\right) x^k (1-x)^{n-k} \\
&= \sum_{k=1}^{n} \binom{n-1}{k-1} \left(\frac{k}{n}\right) x^k (1-x)^{n-k}
\end{aligned}
$$

Change the summation index to $\ell = k-1$, split into two sums, and pull out common factors:

$$
\begin{aligned}
&= \sum_{\ell=0}^{n-1} \binom{n-1}{\ell} \left(\frac{\ell+1}{n}\right) x^{\ell+1} (1-x)^{n-1-\ell} \\
&= \sum_{\ell=0}^{n-1} \binom{n-1}{\ell} \left(\frac{\ell}{n}\right) x^{\ell+1} (1-x)^{n-1-\ell} + \sum_{\ell=0}^{n-1} \binom{n-1}{\ell} \left(\frac{1}{n}\right) x^{\ell+1} (1-x)^{n-1-\ell} \\
&= \frac{(n-1)x}{n} \sum_{\ell=0}^{n-1} \binom{n-1}{\ell} \left(\frac{\ell}{n-1}\right) x^{\ell} (1-x)^{n-1-\ell} + \frac{x}{n} \sum_{\ell=0}^{n-1} \binom{n-1}{\ell} x^{\ell} (1-x)^{n-1-\ell} \\
&= \frac{(n-1)x}{n} \cdot x + \frac{x}{n}
\end{aligned}
$$

(using (*) and (**) above for the last step). This is what we wanted to show.