Mathematics 371 – Numerical Analysis
Midterm Problem Set 2
November 2, 2001

Ground Rules

The problems below form the second midterm problem set announced in the course syllabus. Unless explicitly stated to the contrary, Maple can be used for any portion of any question where it might be useful. This is an open book, open notes, open Swords 219, open Swords 335 assignment. You must work *independently* on these problems, and you may not consult with other faculty members, etc. You may consult other sources in the library, on the Internet, etc. but if you find something relevant there, please *footnote it.* When you have finished your solutions, please write out a short statement that you have abided by these ground rules, sign it, and return it with your solutions.

This assignment is due no later than *5:00pm on Friday, November 9.*

I. (Multivariable Polynomial Interpolation). Suppose we are given $(n+1)(m+1)$ data points of the form (x_i, y_j, z_{ij}) where $0 \le i \le n$, $0 \le j \le m$, and assume the $P_{ij} = (x_i, y_j)$ are distinct points in the xy plane. (Note: the $P_{ij} = (x_i, y_j)$ are points in a rectangular grid in the xy plane, at the intersections of the $n+1$ lines $x = x_i$ parallel to the y-axis and the $m+1$ lines $y = y_j$ parallel to the x-axis.)

A) Consider one particular pair of indices i_0, j_0. Construct a polynomial function of two variables of degree at most n in x and at most m in y that takes the value 1 at $P_{i_0 j_0}$ and 0 at $P_{k\ell}$ for $(k, \ell) \neq (i_0, j_0)$.
B) Show that there exists a unique polynomial $P(x, y)$ in two variables, of degree at most n in x and at most m in y, which interpolates the given data points.
C) Construct the interpolating polynomial as in B for the data points

$$(1, 1, 5), (1, 2, 9), (1.5, 1, 8), (1.5, 2, 10), (2, 1, 0), (2, 2, -1)$$

II. In a chemical experiment, the following data is generated, giving the concentration X of a certain compound in a solution as a function of time:

t:	0	.04	.14	.21	1.22	1.5	1.93
X:	0	.849	1.894	1.986	1.895	1.5	.694

A) Plot the points and interpolate a smooth curve passing through them "by eye".
B) Compute and plot the unique interpolating polynomial of degree ≤ 6 for this data set.
C) Compute and plot a clamped cubic spline interpolating function for this data set. (Note: you will need to estimate the derivatives of X at the endpoints $t = 0$ and $t = 1.93$ in some suitable way. Explain how you are doing that.) Which interpolation method, the one used in part B or the one used in this part, matched your intuitive smooth curve better?

III.

A) Use the Taylor series for $f(x)$ at $x = x_0$ to derive the following 3-point centered difference approximation to $f''(x_0)$ and error term:

$$f''(x_0) = \frac{f(x_0 - h) - 2f(x_0) + f(x_0 + h)}{h^2} - \frac{h^2}{12} f^{(4)}(\xi)$$

for some $\xi \in [x_0 - h, x_0 + h]$.

B) Prove the "Useful Lemma" from class on Wednesday October 31: If F is continuous on $[a, b]$, $c_1, c_2 \in [a, b]$, and $d_1, d_2 \in \mathbf{R}$ satisfy $d_1, d_2 \geq 0$, then there exists some c in $[a, b]$ such that $(d_1 + d_2)F(c) = d_1 F(c_1) + d_2 F(c_2)$.

IV. Determine (with proof) a method that will produce an approximation to

$$\int_0^{48} \sqrt{1 + \cos^2(x)}\, dx$$

accurate to 10^{-4} (at least). Use your method to approximate the value of this integral.