Mathematics 371 – Numerical Analysis
Midterm Problem Set 1 – Selected Solutions
October 6, 2001

E. and F. Both of these problems can be handled in much the same way. Our hypothesis in both is that $f(x)$ has a root of multiplicity m at $x = p$, so

$$f(x) = (x - p)^m h(x),$$

where $lim_{x \to p} h(x) = L \neq 0$. To determine the order of convergence of Newton-Raphson (question E) and the "modified Newton-Raphson" iteration

$$x_n = x_{n-1} - \frac{mf(x_{n-1})}{f'(x_{n-1})}$$

we must analyze $\lim_{x \to p} g'(x)$ for the appropriate iteration function $g(x)$:

$g(x) = x - \frac{f(x)}{f'(x)}$ for Newton-Raphson, and

$g_m(x) = x - \frac{mf(x)}{f'(x)}$ for modified Newton-Raphson.

Note that the term $G(x) = \frac{f(x)}{f'(x)}$ appears in both iteration functions, and

$g'(x) = 1 - G'(x)$ for Newton-Raphson, and

$g'_m(x) = 1 - mG'(x)$ for modified Newton-Raphson.

For $f(x) = (x - p)^m h(x)$, for $x \neq p$, we get:

$$\begin{aligned}
G(x) &= \frac{(x-p)^m h(x)}{m(x-p)^{m-1}h(x) + (x-p)^m h'(x)} \\
&= \frac{(x-p)h(x)}{mh(x) + (x-p)h'(x)} \\
\Rightarrow G'(x) &= \frac{(mh(x) + (x-p)h'(x))(h(x) + (x-p)h'(x))}{(mh(x) + (x-p)h'(x))^2} \\
&\quad - \frac{(x-p)h(x)((m+1)h'(x) + (x-p)h''(x))}{(mh(x) + (x-p)h'(x))^2}
\end{aligned}$$

Hence, $lim_{x \to p} G'(x) = \frac{mL^2}{m^2L^2} = \frac{1}{m}$.

In the Newton-Raphson case (problem E), we get

$$\lim_{x \to p} g'(x) = 1 - \lim_{x \to p} G'(x) = 1 - \frac{1}{m}$$

Since this is not zero when $m \geq 2$, the convergence of Newton-Raphson is (only) linear in these cases. Moreover, the bigger m is, the closer $\lim_{x \to p} g'(x)$ is to 1, so the *slower* convergence will be. Double roots ($m=2$) will give faster convergence than triple roots ($m=3$) and so forth.

In the modified Newton-Raphson case (problem F), we get

$$\lim_{x\to p} g_m'(x) = 1 - m \lim_{x\to p} G'(x) = 1 - m \cdot \frac{1}{m} = 0$$

The convergence of modified Newton-Raphson is quadratic. However, we do *need to know the right m to use in order to set up the correct iteration function to get quadratic convergence.* That information might not be known (or easy to determine), so Steffensen might be superior in many cases.

G. First, by Taylor's theorem, if g''' is continuous, and if $g'(p) = g''(p) = 0$, then for the fixed-point iteration $x_n = g(x_{n-1})$, we have

$$\begin{aligned}
x_n = g(x_{n-1}) &= g(p) + g'(p)(x_{n-1}-p) + \frac{g''(p)}{2}(x_{n-1}-p)^2 + \frac{g'''(\xi)}{6}(x_{n-1}-p)^3 \\
&= p + \frac{g'''(\xi)}{6}(x_{n-1}-p)^3 \\
\Rightarrow |x_n - p| &= \frac{|g'''(\xi)|}{6}|x_{n-1}-p|^3
\end{aligned}$$

which gives cubic convergence (at least).

So the rest of the problem is to show that $g'(p) = g''(p) = 0$ when $f(p) = 0$, $f'(p) \neq 0$, and

$$g(x) = x - \frac{f(x)}{f'(x)} - \frac{1}{2}\frac{f''(x)}{f'(x)}\left(\frac{f(x)}{f'(x)}\right)^2$$

Since $g'''(x)$ will contain $f^{(4)}(x)$, the hypotheses should also include the condition that $f^{(4)}(x)$ is continuous on some interval containing p. Let us first compute $g'(x)$, simplifying whenever possible, using the product, quotient, and chain rules:

$$\begin{aligned}
g' = 1 &- \frac{(f')^2 - ff''}{(f')^2} - \frac{1}{2}\frac{f'f''' - (f'')^2}{(f')^2} \cdot \left(\frac{f}{f'}\right)^2 \\
&- \frac{f''}{f'} \cdot \frac{f}{f'} \cdot \frac{(f')^2 - ff''}{(f')^2} \\
= &\ \frac{2ff''(f')^2 - f^2f'f''' + f^2(f'')^2 - 2ff''(f')^2 + 2f^2(f'')^2}{2(f')^4} \\
= &\ \frac{3f^2(f'')^2 - f^2f'f''}{2(f')^4}
\end{aligned}$$

Let's try a bit of "creative laziness" here. Note that both terms on the top have a factor of f^2 so we can write $g'(x) = (f(x))^2 H(x)$. Since we assume $f(p) = 0$ and $f'(p) \neq 0$, we get $g'(p) = 0$. Moreover, if we differentiate again,

$$g''(x) = 2f(x)f'(x)H(x) + (f(x))^2 H'(x)$$

and $g''(p) = 0$ also since both terms still contain a factor of $f(x)$, while the denominators in H, H' contain only powers of f'. This completes the proof that $g'(p) = g''(p) = 0$.