Mathematics 371 – Numerical Analysis
Selected Solutions – Midterm Problem Set 2
November 14, 2001

I.

A) The desired polynomial can be constructed using one-variable Lagrange polynomials:

$$\begin{aligned} Q_{i,j}(x,y) &= L_{n,i}(x)L_{m,j}(y) \\ &= \frac{\prod_{p\neq i}(x-x_p)}{\prod_{p\neq i}(x_i-x_p)} \cdot \frac{\prod_{q\neq j}(y-y_q)}{\prod_{q\neq j}(y_j-y_q)} \end{aligned}$$

Note that $Q(x_k, y_\ell)$ is zero unless $k = i$ and $\ell = j$ (since at least one factor in the numerator of the product is zero), and $Q(x_i, y_j) = 1$.

B) *Existence:* Given $z_{i,j}$, consider the polynomial

$$\begin{aligned} p(x,y) &= \sum_{i=0}^{n}\sum_{j=0}^{m} z_{i,j} Q_{i,j}(x,y) \\ &= \sum_{i=0}^{n}\sum_{j=0}^{m} z_{i,j} L_{n,i}(x) L_{m,j}(y) \end{aligned}$$

Then by part A, we know that for all (k, ℓ)

$$p(x_k, y_\ell) = \sum_{i=0}^{n}\sum_{j=0}^{m} z_{i,j} Q_{i,j}(x_k, y_\ell) = 0 + \cdots + 0 + z_{k,\ell}\cdot 1 + 0 + \cdots + 0.$$

Since each $Q_{i,j}(x,y)$ is of degree $\le n$ in x and degree $\le m$ in y, the same is true for $p(x,y)$. This proves existence for a polynomial satisfying the conditions given.

Uniqueness: To prove that $p(x,y)$ is unique, suppose we have two polynomials $p_1(x,y)$ and $p_2(x,y)$ satisfying $p_1(x_i, y_j) = p_2(x_i, y_j) = z_{i,j}$ for all $0 \le i \le n$ and $0 \le j \le m$. Then the polynomial $p(x,y) = p_1(x,y) - p_2(x,y)$ has degree $\le n$ in x and degree $\le m$ in y and is zero at the $(n+1)(m+1)$ points (x_i, y_j). We can collect powers of y to write:

$$p(x,y) = c_m(x)y^m + c_{m-1}(x)y^{m-1} + \cdots + c_1(x)y + c_0(x). \tag{1}$$

Each coefficient polynomial $c_p(x)$ has degree $\le n$ in x, since $p(x,y)$ has degree $\le n$ in x.

If we substitute $x = x_i$, any $0 \le i \le n$ into (1), we obtain

$$p(x_i, y) = c_m(x_i)y^m + c_{m-1}(x_i)y^{m-1} + \cdots + c_1(x_i)y + c_0(x_i)$$

This is a polynomial of degree $\leq m$ in y that has $m+1$ roots $y_0, y_1, \ldots, y_m$. Hence it must be identically zero, or $c_m(x_i) = 0, \ldots, c_0(x_i) = 0$. But this is true for all x_i so each of the $c_p(x)$ are polynomials of degree $\leq n$ in x with $n+1$ distinct roots x_i. Therefore all of the $c_p(x)$ are identically zero. Hence $p(x, y)$ is also identically zero, so $p_1(x, y) = p_2(x, y)$.

III.

A) The Taylor series for f looks like:

$$f(x) = f(x_0)+f'(x_0)(x-x_0)+\frac{f''(x_0)}{2!}(x-x_0)^2+\frac{f'''(x_0)}{3!}(x-x_0)^3+\frac{f^{(4)}(\xi(x))}{4!}(x-x_0)^4$$

So at $x = x_0 + h$ and $x = x_0 - h$, we get

$$f(x_0+h) = f(x_0) + f'(x_0)h + \frac{f''(x_0)}{2!}h^2 + \frac{f'''(x_0)}{3!}h^3 + \frac{f^{(4)}(\xi(x_0+h))}{4!}h^4$$
$$f(x_0-h) = f(x_0) - f'(x_0)h + \frac{f''(x_0)}{2!}h^2 - \frac{f'''(x_0)}{3!}h^3 + \frac{f^{(4)}(\xi(x_0-h))}{4!}h^4$$

Add these last two equations to get:

$$f(x_0+h) + f(x_0-h) = 2f(x_0) + f''(x_0)h^2 + \frac{1}{24}(f^{(4)}(\xi_1) + f^{(4)}(\xi_2))h^4$$

where $\xi_1 = \xi(x_0+h)$ and $\xi_2 = \xi(x_0-h)$. Rearranging yields

$$f''(x_0) = \frac{f(x_0+h) - 2f(x_0) + f(x_0-h)}{h^2} - \frac{1}{24}(f^{(4)}(\xi_1) + f^{(4)}(\xi_2))h^2.$$

To finish, we need to assume that $f^{(4)}$ is continuous on an interval containing $[x_0 - h, x_0 + h]$. Then the "Useful Lemma" (part B) implies that

$$\frac{1}{24}(f^{(4)}(\xi_1) + f^{(4)}(\xi_2))h^2 = \frac{1}{12}f^{(4)}(\xi)h^2$$

for some $\xi \in [x_0 - h, x_0 + h]$.

B) Consider

$$\frac{d_1}{d_1+d_2}F(c_1) + \frac{d_2}{d_1+d_2}F(c_2)$$

By the standard properties of continuous functions, F has a maximum M and a minimum m on the interval $[a, b]$. Hence

$$\frac{d_1}{d_1+d_2}F(c_1) + \frac{d_2}{d_1+d_2}F(c_2) \leq \frac{d_1}{d_1+d_2}M + \frac{d_2}{d_1+d_2}M = M$$

Similarly,

$$\frac{d_1}{d_1+d_2}F(c_1)+\frac{d_2}{d_1+d_2}F(c_2)\geq\frac{d_1}{d_1+d_2}m+\frac{d_2}{d_1+d_2}m=m$$

Hence for all c_1, c_2,

$$m\leq\frac{d_1}{d_1+d_2}F(c_1)+\frac{d_2}{d_1+d_2}F(c_2)\leq M$$

Then by the Intermediate Value Theorem, there exists a $c\in[a,b]$ such that

$$F(c)=\frac{d_1}{d_1+d_2}F(c_1)+\frac{d_2}{d_1+d_2}F(c_2)$$

which is equivalent to what we wanted to show.