Mathematics 371 – Numerical Analysis
Lab 4 – Adaptive Quadrature and Gaussian Quadrature
November 12, 2001

Goals

In this lab we will use the *adaptive quadrature* methods introduced in class last week, and continue the development of *Gaussian quadrature*.

Adaptive Quadrature

A) The implementation of the recursive adaptive quadrature procedure `Adapt` we discussed in class on Wednesday, November 7 is available on the course homepage. Download it, save to a file in your home directory, and read it into your Maple session. Use it to approximate the integrals:

$$\int_0^2 \sin(x^3)\ dx, \qquad \int_1^{10} \frac{1-\cos(x)}{x^2}\ dx$$

accurate to 10^{-5}. Compare with Maple's approximate value obtained using a command of the form

```
evalf(Int(f(x),x = a..b);
```

B) (For after the lab) Draw the "recursion tree" of calls to the `Adapt` procedure on either one of the integrals from part A – your choice – by having the procedure print out the tracing messages. Discuss what you see there relative to the graph of the appropriate function on the interval of integration.

Gaussian Quadrature

Recall from class last Friday that the Gaussian quadrature method on an interval $[a, b]$ depends on knowing an *orthogonal basis* for the vector space of polynomial functions on $[a, b]$ of degree $\leq n$ for some n – that is, a collection of polynomials

$$\{p_0(x), p_1(x), \ldots, p_n(x)\}$$

with the properties that

a) (orthogonality)

$$\langle p_i, p_j \rangle = \int_a^b p_i(x)p_j(x)\ dx = \begin{cases} 0 & \text{if } i \neq j, \text{ and} \\ c > 0 & \text{if } i = j \end{cases}$$

b) $\deg p_i(x) = i$ for all $i = 0, \ldots, n$.
c) The equation $p_i(x) = 0$ has i distinct real roots in the interval $[a, b]$ for all $i = 0, \ldots, n$.

Knowing such a collection of polynomials, and writing $x_1, \ldots, x_n$ for the roots of the equation $p_n(x) = 0$ on $[a, b]$, we showed that the numerical integration rule:

$$(1) \qquad \int_a^b f(x)\,dx = \sum_{j=1}^{n} c_j f(x_j)$$

(where

$$(2) \qquad c_j = \int_a^b \frac{\prod_{k \neq j}(x - x_k)}{\prod_{k \neq j}(x_j - x_k)}\,dx$$

for each $j = 1, \ldots n$) has degree of precision $2n - 1$(!)

C) In this problem, you will use Maple to construct a collection of such polynomials on $[a, b] = [-1, 1]$ “from scratch.”

1) Check that $p_0(x) = 1$ and $p_1(x) = x$ are orthogonal under the inner product

$$\langle f, g \rangle = \int_{-1}^{1} f(x)g(x)\,dx.$$

Note that $p_0(1) = p_1(1) = 1$ also.

2) Use the following outline to construct a polynomial p_2 of degree 2 that is orthogonal to both p_0 and p_1. First, let $p_2(x) = ax^2 + bx + c$ with undetermined (variable) coefficients. Use Maple to compute $\int_{-1}^{1} p_0(x)p_2(x)\,dx$ and $\int_{-1}^{1} p_1(x)p_2(x)\,dx$. Setting your results equal to zero yields two linear equations for a, b, c. Solve these together with the equation obtained from $p_2(1) = 1$ to determine a, b, c. Maple note: To solve a system of linear equations in Maple, you can use a command of the following form.

```
solve(eqns,vars);
```

where `eqns` is the set of equations to be solved, and `vars` is the set of variables you want to solve for. For example, to solve

$$3x - 2y = 1$$
$$4x + 5y = 7$$

you can use

```
solve({3*x - 2*y = 1, 4*x + 5*y = 7},{x,y});
```

3) By similar computations, now compute polynomials $p_3(x), p_4(x), p_5(x)$ (in that order) so that $\{p_0, p_1, p_2, p_3, p_4, p_5\}$ is orthogonal, $p_i(1) = 1$, and $\deg p_i = i$ for all i.

4) The polynomials you are computing here are called the *Legendre polynomials*. Maple has a package that computes these by a different method, so you can use this to check your work from parts 2 and 3: First execute the command

`with(orthopoly);`

The nth Legendre polynomial $p_n(x)$ is written as `P(n,x)` in Maple. List out `P(0,x)` through `P(5,x)` and verify that your results from 2 and 3 are correct.

5) Find the roots of each of $p_0, \ldots, p_5$ using the `complex` option in the `fsolve` command (consult the Maple online help listings for the syntax of this one).
6) Compute the weights c_j, $j = 1, \ldots, n$ from (1) using the roots of each of p_i, $i = 2, \ldots 5$. (There is one set of such weights for each i.)
7) Use Gaussian quadrature with $n = 2, 3, 4, 5$ to approximate the integrals

$$\int_{-1}^{1} \cos(\sin(x) - 4)\ dx$$

and

$$\int_{-1}^{1} \exp(-x^2)\ dx$$

and compare with Maple's approximations for these integrals (output from commands `evalf(Int(f(x),x=-1..1));`). What is happening as n increases?

8) How would you use Gaussian quadrature to approximate $\int_a^b f(x)\ dx$ if $[a, b] \neq [-1, 1]$? Explain and use your method to approximate

$$\int_{1}^{2} \frac{\cos^2(x)}{x}\ dx$$

D) (For after the lab) In this problem, you will prove that the Legendre polynomial $p_n(x)$ actually has n distinct real roots in $[-1, 1]$ for all n.

1) First note that $\int_{-1}^{1} p_n(x)\ dx = 0$ (why?). Use this to deduce that $p_n(x) = 0$ has at least one solution in $[-1, 1]$.
2) Aiming for a contradiction, suppose that p_n changes sign only some number $1 \leq j < n$ times in $[-1, 1]$, say at $r_1, \ldots, r_j$. What can you say about

$$\int_{-1}^{1} (r_1 - x)(r_2 - x) \cdots (r_j - x) p_n(x)\ dx?$$

Show that this leads to a contradiction to the orthogonality of $\{p_0, p_1, \ldots, p_n\}$.

E) (Also for after the lab) Problem 7 from Section 4.7 in Burden and Faires.

Assignment

Due: In class, Monday, November 19.