Mathematics 371 – Numerical Analysis
Lab Day 3 – Using Cubic Spline Interpolation
October 19, 2001

Goals and Background

The goals of today's lab are to use the Maple `Spline` procedure for *free cubic splines* and another "homegrown" Maple procedure for *clamped cubic splines*. We will use them to construct cubic spline interpolating functions. Recall that on Wednesday we saw that given $x_0, \ldots, x_n$ and function values $f(x_0), \ldots, f(x_n)$ to be interpolated, the coefficients a_j, b_j, c_j, d_j in the free cubic spline interpolating function

$$S(x) = \begin{cases} S_0(x) = a_0 + b_0(x - x_0) + c_0(x - x_0)^2 + d_0(x - x_0)^3, \ x_0 \le x \le x_1 \\ S_1(x) = a_1 + b_1(x - x_1) + c_1(x - x_1)^2 + d_1(x - x_1)^3, \ x_1 \le x \le x_2 \\ \quad \vdots \\ S_{n-1}(x) = a_{n-1} + b_{n-1}(x - x_{n-1}) + c_{n-1}(x - x_{n-1})^2 + d_{n-1}(x - x_{n-1})^3, \\ \qquad\qquad x_{n-1} \le x \le x_n \end{cases}$$

satisfy $a_j = f(x_j)$ for $0 \le j \le n-1$ and

$$\begin{aligned} b_j &= \frac{a_{j+1} - a_j}{h_j} - \frac{h_j}{3}(2c_j + c_{j+1}) \\ d_j &= \frac{c_{j+1} - c_j}{3h_j} \end{aligned}$$

for all $j = 0, \ldots, n-1$. The c_j are found by solving the $(n-1) \times (n-1)$ system of linear equations:

$$\begin{pmatrix} 2(h_0 + h_1) & h_1 & 0 & \cdots & 0 & 0 \\ h_1 & 2(h_1 + h_2) & h_2 & \cdots & 0 & 0 \\ \vdots & & & & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & h_{n-2} & 2(h_{n-2} + h_{n-1}) \end{pmatrix} \cdot \begin{pmatrix} c_1 \\ \vdots \\ c_{n-1} \end{pmatrix}$$
$$= \begin{pmatrix} \frac{3(a_2 - a_1)}{h_1} - \frac{3(a_1 - a_0)}{h_0} \\ \vdots \\ \frac{3(a_n - a_{n-1})}{h_{n-1}} - \frac{3(a_{n-2} - a_{n-1})}{h_{n-1}} \end{pmatrix}$$

where $h_j = x_{j+1} - x_j$ and $c_0 = c_n = 0$. There is a very similiar system of linear equations whose solutions give the c_j coefficients for clamped cubic splines.

A New Maple Command

As we saw in the handout from Monday's class, Maple has a spline interpolation function that can be used to construct free cubic spline interpolating functions. It is invoked by first entering the command:

```
with(CurveFitting);
```

Then the format of the spline command is similar to the `PolynomialInterpolation` command from the last problem set.

```
Spline(xlist,ylist,var);
```

where `xlist` is the list of x-coordinates of the interpolation points (that is, the x_j), and `ylist` is the list of y-coordinates $f(x_j)$. These are Maple lists so, they should be lists of floating point numbers, separated by commas in *square brackets*: [,]. The output from the spline command is the piecewise cubic definition of the *natural cubic spline* function $S(x)$ – in Maple terms it is an *expression, not a function*, though(!) You make it into a function, assign it to a symbolic name, plot it, or whatever you like.

Laboratory Questions

A) In class on Wednesday, we worked out an example where $x_0 = 0, x_1 = 1, x_2 = 3, x_3 = 4$ and $f(x_0) = a_0 = 5, f(x_1) = a_1 = 4, f(x_2) = a_2 = 0, f(x_3) = a_3 = 1$. We found that the cubic spline was given by

$$\begin{cases} S_0(x) = 5 - \frac{5}{8}x - \frac{3}{8}x^3, & 0 \leq x \leq 1 \\ S_1(x) = 4 - \frac{7}{4}(x-1) - \frac{9}{8}(x-1)^2 + \frac{1}{2}(x-1)^3, & 1 \leq x \leq 3 \\ S_2(x) = -\frac{1}{4}(x-3) + \frac{15}{8}(x-3)^2 - \frac{5}{8}(x-3)^3, & 3 \leq x \leq 4 \end{cases}$$

Use the `Spline` command to check this work, plot $S(x)$ and the points

$$(0,5), (1,4), (3,0), (4,1)$$

together. See the Maple online help page for the `plot` command for some ideas about plotting sets of points (in particular, the option `style = point`). You will probably want to make the combined plot of $S(x)$ and the points by using the `display` command from the `plots` package: Generate the two plots separately and assign them to symbolic names (this is one case where you probably *don't* want to see the output, so use a colon at the end of the plot commands), then use `display` combine them.

B) Use Maple's `Spline` command to do problems 3d and 4d in section 3.4 of Burden and Faires. Also, plot the spline together with $f(x) = x \cos x - 2x^2 + 3x - 1$, and discuss the goodness of the "fit" between the spline function and $f(x)$.

C) Use Maple's `Spline` command to do problem 27 in section 3.4 of Burden and Faires. Note that this is the same collection of data points as you used in problem 25 from section 3.1 on the last problem set, but there you used interpolating polynomials. Compare and contrast your results using interpolating polynomials and interpolating splines.

D) Maple does not have a built-in procedure or a library package for computing *clamped* cubic splines. However, you can get the source code for one on the course homepage. Do this and read the code into your Maple session. The format for using the procedure is similar to the one for `spline`, except you also need to supply the values for the derivative of the spline function at the endpoints (`df0, dfn` are $f'(x_0)$ and $f'(x_n)$ respectively):

```
clampedspline(xlist,ylist,df0,dfn,var);
```

Use this procedure to solve problem 28 in section 3.4 of Burden and Faires.

Assignment

Individual lab write-ups due in class Friday, October 26.