Mathematics 371 – Numerical Analysis
Discussion 2 – Fixed Point Iteration, Theory
September 14, 2001

Background

Recall that we have introduced the idea of *fixed point iteration* as a method to solve equations $f(x) = 0$. We first *rearrange the equation* to the form $x = g(x)$ using "reversible" operations so that $f(x) = 0 \Leftrightarrow x = g(x)$, then attempt to find a point p satisfying $p = g(p)$ by starting at some x_0 and iterating g to construct the sequence:

$$\begin{aligned} x_1 &= g(x_0) \\ x_2 &= g(x_1) \\ x_3 &= g(x_2) \\ &\vdots \\ x_{n+1} &= g(x_n) \end{aligned} \tag{1}$$

You can visualize an iteration sequence this way: Draw the graph $y = g(x)$ together with the line $y = x$. Starting from $x = x_0$ (on the x-axis), draw a vertical line segment up or down to the point $(x_0, g(x_0))$ on the graph $y = g(x)$. Then connect that point to the point on the line $y = x$ with the same y-coordinate. The vertical line through that point intersects the x-axis in $x_1 = g(x_0)$ (why?). Then repeat the process to get $x_2 = g(x_1)$, etc.

Also recall the "Most Valuable Theorem" of differential calculus:

Mean Value Theorem. *Let $f(x)$ be continuous on $[a, b]$ and differentiable on (a, b). Then there exists a $c \in (a, b)$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$.*

In MATH 242, you saw how the MVT establishes connections between properties of $f'(x)$ (such as the sign of the derivative), and properties of the function $f(x)$ (such as whether f is increasing or decreasing). Today, we will see another connection dealing with *fixed point iteration* for numerical root-finding.

Discussion Questions

A) If an iteration sequence x_n constructed using g as in (1) above converges to some $q \in \mathbf{R}$, must q satisfy $g(q) = q$? What condition on g do you need to assume to say this?

B) From the examples you have seen, given a function $g(x)$ and a fixed point p of g, when does the iteration starting from x_0 close to p yield a sequence x_n converging to p? Make a conjecture (an "educated guess") about a general statement along these lines.

C) If we assume that $g(x)$ is differentiable on an interval containing a fixed point p, the Mean Value Theorem can be used to prove a statement guaranteeing convergence to the fixed point. Think about the following question: Given $q = g(q)$ and x_0 near q, if $x_1 = g(x_0)$, then by the MVT,

$$|x_1 - p| = |g(x_0) - g(p)| = |g'(c)||x_0 - q|$$

for some $c \in (x_0, p)$. When is it guaranteed that x_1 is *closer to p than x_0 was?* Then consider subsequent steps in the iteration, and prove your statement from B, or perhaps a modification of what you originally thought.

D) What can you say about the *order of convergence* of fixed point iteration? If $x_n \to p$ is a convergent iteration sequence, for what function $h(n)$ is $x_n = p + O(h(n))$? Explain.

Assignment

Group write-up, due Wednesday, September 19.