Mathematics 371 – Numerical Analysis
Discussion 1 – Rates of Convergence
September 5, 2001

Background

Many numerical algorithms are *iterative* – that is, they involve executing some sequence of computations repeatedly using the results of the previous stage to derive the result of the current stage in the process. In the nth stage, we usually have a current approximation to the quantity we are trying to compute. If we call that current approximation x_n and the desired quantity $x^\star$, then in principle (i.e. if we never stopped the process), we would generate a *sequence of approximations* x_n converging to $x^\star$ as $n \to \infty$. In this situation, there is a notion of *rate of convergence* that can be used to describe *how fast x_n approaches $x^\star$*:

We say $x_n = x^\star + O(n^{-p})$, or $x_n \to x^\star$ with rate of convergence $O(n^{-p})$, if there is a positive constant K and $n_0 \geq 1$ such that $|x_n - x^\star| \leq K \cdot \frac{1}{n^p}$ for all $n \geq n_0$.

Discussion Questions

A)

1) The two sequences

$$x_n = 1 + \frac{1}{10n} \qquad y_n = 1 + \frac{(-1)^n}{n^2}$$

both approach $x^\star = 1$ as $n \to \infty$. Intuitively, which approaches its limit *faster*? Explain by analyzing $|x_n - 1|$ and $|y_n - 1|$.

2) What is the *largest* power p such that $x_n = 1 + O(n^{-p})$? Same question for y_n? How does this relate to your answer to part 1?

B) (One not-so-intuitive feature of our definition) Let $0 < q < p$ and assume that $x_n = x^\star + O(n^{-p})$. Then show that $x_n = x^\star + O(n^{-q})$ as well.

C) Suppose $x_n = f(\frac{1}{n})$ is produced by evaluating some differentiable function f. How could you use properties of that function to analyze the rate of convergence of x_n? Use your idea to determine the rate of convergence of each of the following sequences.

1) $\lim_{n\to\infty} \sin\left(\frac{1}{n}\right) = 0$
2) $\lim_{n\to\infty} n \sin\left(\frac{1}{n}\right) = 1$ (first show that this sequence does tend to 1 as $n \to \infty$)
3) $\lim_{n\to\infty} (\ln(n+1) - \ln(n)) = 0$ (first show that this sequence does tend to 0 as $n \to \infty$)
4) $\lim_{n\to\infty} n(1 - e^{1/n})$ (what is the limit?)

D) In some very good cases, we will be able to produce numerical algorithms that converge to their results with a rate that is $O(n^{-p})$ for *all* $p \geq 1$ (no matter how large). This means we have a *very fast* rate of convergence.

1) For instance show that $x_n = 10^{-2^n} \to 0$ has this property: $10^{-2^n} = O(n^{-p})$ for all $p \geq 1$.
2) A sequence $x_n \to x^\star$ is said to *converge quadratically (respectively cubically)* to its limit if there exists a positive constant K such that

$$\lim_{n\to\infty} \frac{|x_{n+1} - x^\star|}{|x_n - x^\star|^2} = K,$$

respectively

$$\lim_{n\to\infty} \frac{|x_{n+1} - x^\star|}{|x_n - x^\star|^3} = K.$$

(In intuitive terms, quadratic convergence, for instance, means that for large n,

$$|x_{n+1} - x^\star| \doteq K|x_n - x^\star|^2.$$

Show that $x_n = 10^{-2^n} \to 0$ has quadratic convergence.
3) Construct a sequence that converges to 0 cubically.

Assignment

Write-ups due Monday, September 10.