Mathematics 41 – Multivariable Calculus
Vector Calculus – The Essentials

0. Introduction

We did not have time this semester to give a satisfactory treatment of one final topic in the syllabus for the course. These notes will serve as a reference for the ideas behind line integrals and Green's Theorem, if you ever run into them in courses you take in the future.

1. Path Integrals, Line Integrals and Green's Theorem

Let $\alpha : [a, b] \to D$ be a smooth parametric curve in the plane. Let $T(t) = \alpha'(t)/\|\alpha'(t)\|$ be its unit tangent vector, and write $ds = \|\alpha'(t)\|\, dt$ for the arclength element on α (that is, $\int_a^b ds$ computes the arclength along the curve from $t = a$ to $t = b$. If $f(x, y)$ is a reasonably nice (e.g. continuous) *scalar* function defined on D, we can define:

Definition 1. *The path integral of f over α is the integral*

$$\int f ds = \int_a^b f(\alpha(t))\|\alpha'(t)\|\, dt$$

Example 1. For example, let $f = 3x + 2y$, and let $\alpha : [0, \pi] \to \mathbf{R}^2$ be defined by $\alpha(t) = (\cos(t), \sin(t))$ (the upper semicircle of the circle of radius 1 centered at $(0, 0)$). The arclength element of this curve is

$$ds = \|\alpha'(t)\| = \|(-\sin(t), \cos(t))\|\, dt = \sqrt{\sin^2(t) + \cos^2(t)}\, dt = dt,$$

Then the path integral of f over the curve α is

$$\int_\alpha f ds = \int_0^\pi 3\cos(t) + 2\sin(t)\, dt = [3\sin(t) - 2\cos(t)]_0^\pi = 4$$

A closely related concept is the *line integral of a vector field.* The best way to distinguish these in your mind is to remember that path integrals are defined for scalar functions, while line integrals are defined for vector functions (that is, vector fields).

Let $\mathbf{F}(x, y) = (P(x, y), Q(x, y))$ be a continuous vector field on a region $D \subset \mathbf{R}^2$. In applications, $\mathbf{F}(x, y)$ might represent a force field, the velocity field for a moving fluid, etc. Let $\alpha : [a, b] \to D$ is a smooth parametric curve whose image lies in D, with unit tangent T as above.

Definition 2. *We define the line integral of* $\mathbf{F}$ *along α, denoted*

$$\int_\alpha \mathbf{F} \cdot T\, ds = \int_\alpha \mathbf{F} \cdot \alpha'\, dt$$

as follows:

$$\int_\alpha \mathbf{F}\cdot T\,ds = \int_a^b \mathbf{F}(\alpha(t))\cdot\alpha'(t)\,dt$$

The notation $\mathbf{F}(\alpha(t))$ means this: If $\alpha(t) = (x(t), y(t))$ is the parametrization, and $\mathbf{F} = (P(x,y), Q(x,y))$, then $\mathbf{F}(\alpha(t)) = (P(x(t), y(t)), Q(x(t), y(t)))$. We substitute the components of the parametrization for x, y in $\mathbf{F}$.

Note that the line integral of a vector field $\mathbf{F}$ could also be defined as the path integral of the scalar function $\mathbf{F}\cdot T$ (the component of $\mathbf{F}$ in the direction of the unit tangent to the curve).

In physics, if $\mathbf{F}$ represents a *force* applied to an object moving along the path α, then $\int_\alpha \mathbf{F}\cdot T\,ds$ represents the *total work done* (the integral of the component of $\mathbf{F}$ in the direction of the path).

For computing line integrals, we usually use the observation that the factors of $\|\alpha'(t)\|$ in the denominator from T, and in the numerator from ds always cancel. Hence there is no need to compute T or ds, and we almost always simply compute $\int_a^b \mathbf{F}(\alpha(t))\cdot\alpha'(t)\,dt$.

Example 2. Let α be the curve from Example 1. If $\mathbf{F}$ is the vector field $\mathbf{F} = (x^2, y+1)$, then the line integral of $\mathbf{F}$ along α is equal to

$$\begin{aligned}
\int_0^\pi \mathbf{F}(\alpha(t))\cdot\alpha'(t)\,dt &= \int_0^\pi (\cos^2(t), \sin(t)+1)\cdot(-\sin(t), \cos(t))\,dt \\
&= \int_0^\pi -\cos^2(t)\sin(t) + \sin(t)\cos(t) + \cos(t)\,dt \\
&= [\cos^3(t)/3 - \cos^2(t)/2 + \sin(t)]_0^\pi \\
&= -2/3
\end{aligned}$$

A simple closed curve is smooth curve α where $\alpha(a) = \alpha(b)$ but α is one-to-one otherwise on the domain $[a, b]$. This means, intuitively, that α does not cross itself, except that the final point $\alpha(b)$ is the same as $\alpha(a)$. A simple closed curve divides the plane into an interior and an exterior region. Green's Theorem gives an equality between the line integral of a vector field over a simple closed curve and a related double integral over the interior region.

Green's Theorem. *Let $\mathbf{F}(x,y) = (P(x,y), Q(x,y))$ be a vector field whose components have continuous first order partial derivatives on an open $U \subset \mathbf{R}^2$ containing the simple closed curve α and the region D interior to α. If α is traced so that the region D is always to the left as we proceed along α, then*

$$\int_\alpha \mathbf{F}\cdot T\,ds = \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,dA \tag{1}$$

(where dA represents the area element in the plane). If α is traced in the other direction, then a minus sign must be inserted on one side of this equation.

Example 3. For instance, let $\mathbf{F} = (-y, x)$, and let $\alpha = (5\cos(t), \sin(t))$, $t \in [0, 2\pi]$. Here α is an ellipse (semi-major axis = 5, semi-minor axis = 1, center at $(0,0)$, traced counterclockwise. Hence the condition that α is traced so that the region D is always to the left *is satisfied*. Let us verify that the conclusion (1) of Green's Theorem is true here. The line integral

$$\int_\alpha \mathbf{F} \cdot T\, ds = \int_0^{2\pi} 5\sin^2(t) + 5\cos^2(t)\, dt = 10\pi$$

The double integral is

$$\iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dA = \int_{-5}^{5} \int_{-\sqrt{1-x^2/25}}^{\sqrt{1-x^2/25}} 2\, dy\, dx$$

which equals 2 times the area of the ellipse. Recalling the area formula $A = \pi ab$ for an ellipse with semi-major axis a and semi-minor axis b, this equals $2 \cdot \pi \cdot 5 \cdot 1 = 10\pi$, which agrees with the line integral calculation.

Green's Theorem can be used in several ways. First, the relation given in the statement is an important step in many proofs about line and double integrals in the plane. Second, to compute a line integral around a simple closed curve, it may sometimes be simpler to apply Green's Theorem and compute $\iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dA$ instead.

Example 4. For instance, consider the vector field

$$\mathbf{F} = (P, Q) = (x^2 + y, x + y^3)$$

Note that $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1 - 1 = 0$. Hence, *the line integral of* $\mathbf{F}$ *around any simple closed curve is* **zero**, by Green's Theorem. In fact, note $\mathbf{F}$ is a gradient vector field: $\mathbf{F} = \nabla f$, where $f = x^3/3 + xy + y^4/4$. If $\mathbf{F} = \nabla f$ is any gradient vector field, then Green's Theorem implies

$$\int_\alpha \nabla f \cdot T ds = \iint_D \frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x}\, dA$$

If f has continuous second order partial derivatives, then the mixed second order partials are *equal*, so the integrand is zero and the line integral is zero.

In fact, even without using Green's Theorem, it can be seen that:

Proposition. *Let* $\mathbf{F}$ *be the gradient vector field of a function* f *with continuous second order partial derivatives on* U*. Then the line integral of* $\mathbf{F}$ *around every simple closed curve in* U *is zero.*

Proof: By the Chain Rule for partial derivatives,

$$\int_\alpha \nabla f \cdot T\, ds = \int_a^b \nabla f(\alpha) \cdot \alpha'(t)\, dt = \int_a^b \frac{d}{dt} f(\alpha(t)) dt = f(\alpha(b)) - f(\alpha(a))$$

Since α is a simple closed curve the endpoints are the same, and the integral is zero.//

It also follows by the same reasoning that in this case, the line integral of $\mathbf{F}$ *depends only on the endpoints of the path of integration*, not on the actual curve α. $\mathbf{F}$ is said to be a *conservative* vector field if $\mathbf{F} = \nabla f$ for some scalar function f. If $\mathbf{F}$ is conservative, then f is called a *potential function* for $\mathbf{F}$.

There is also a partial converse of the proposition, which states that if $\mathbf{F}$ is continuous on a *simply connected* region (intuitively, a region with "no holes") and the line integrals of $\mathbf{F}$ around all simple closed curves are zero, then $\mathbf{F}$ is a conservative vector field: $\mathbf{F} = \nabla f$ for some potential function f.

2. An Alternate Form of Green's Theorem

Using our notation, this result reads as follows: If α is a simple closed curve, then

$$\int_\alpha \mathbf{F} \cdot \mathbf{N}\, ds = \iint_D \nabla \cdot \mathbf{F}\, dA, \tag{3}$$

where $\mathbf{N}$ is now the outward unit normal vector to the curve α, and $\nabla \cdot \mathbf{F}$ is the "2D-divergence" of $\mathbf{F}$: If $\mathbf{F} = (P(x, y), Q(x, y))$, then

$$\nabla \cdot \mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}.$$

Let's see the connection between (3) and the original statement of Green's Theorem above, (1).

First, we note that if $\alpha(t) = (x(t), y(t))$, then the vector $(y'(t), -x'(t))$ is orthogonal to the tangent vector $\alpha'(t)$. (It is obtained by rotating $\alpha'(t)$ by $\pi/2$ clockwise.) Hence as our unit normal we can take

$$\mathbf{N} = \frac{(y'(t), -x'(t))}{\|(y'(t), -x'(t))\|} = \frac{(y'(t), -x'(t))}{\|\alpha'(t)\|}.$$

(Whether $\mathbf{N}$ points outward or inward for a simple closed curve depends on the parametrization we started from.)

Then, using this $\mathbf{N}$, we have

$$\int_\alpha \mathbf{F} \cdot \mathbf{N}\, ds = \int_a^b P(x(t), y(t))y'(t) - Q(x(t), y(t))x(t)\, dt$$

Note that this is the same thing as we would get for the line integral of

$$\mathbf{G} = (-Q(x, y), P(x, y))$$

around α. But then by the usual form (1) of Green's Theorem, if α is traversed so that D lies to the left, we have:

$$\int_\alpha \mathbf{F} \cdot \mathbf{N}\, ds = \int_\alpha \mathbf{G} \cdot T\, ds = \iint_D \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\, dA = \iint_D \nabla \cdot \mathbf{F}\, dA$$