Mathematics 41, section 1 – Multivariable Calculus
Lab 4 – Applications of Partial Derivatives
October 7, 1999

Goals

Today, we will study how Maple can be used to compute partial derivatives of functions of several variables, and we will look at two examples of *partial differential equations* (or "PDE") that are used to model real-world phenomena such as *heat conduction*, and *vibrations of a string*.

New Maple

In addition to the graphing routines we have used so far, Maple also contains extensive facilities for symbolic computation, including computation of partial derivatives of functions defined by expressions. The command for this is called `diff`. The format is

```
diff(expression,variable);
```

For instance, if we have a function of two variables defined by a command such as the following:

```
f := (x,y) -> cos(x*y^2 - exp(sin(x)));
```

then we can compute its first order partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ using the commands `diff(f(x,y),x);` and `diff(f(x,y),y);`, respectively. Try these and verify the results by hand.

You can also use `diff` to compute second- and higher-order partial derivatives. The "brute force" method for this is by "nesting". For example, to compute $\frac{\partial^2 f}{\partial x^2}$, we could compute `diff(diff(f(x,y),x),x);`, since

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right).$$

There is also a shortcut method for computing higher order derivatives. you can get more information on that by consulting Maple's online help pages (use: Help/Topic Search on "diff".)

Lab Problem A

Use the Maple `diff` command to find all the first and second order partials of each of the following functions. Verify the results by computing the partial derivatives by hand and comparing with Maple's output. If Maple simplifies its result to a different form, explain how your answer and Maple's are the same.

1) $f(x,y) = \arctan(xy) - e^{-(x^2+y^2)}$.
2) $f(x,y) = \cos(\sin(x/y) - xy^2)$

3) $f(x, y) = \ln(x - 4^{xy})$

Partial Differential Equations – Heat Conduction and Vibrations of Strings

Let $u(x, t)$ represent the temperature at position x, and time t in a long thin wire extending from $x = 0$ to $x = L$. Assuming that the lateral surface of the wire is insulated, that the material of the wire has constant thermal properties, and that there are no heat sources in the wire, then it can be shown that the temperature function satisfies the following relation, called the *heat equation*:

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}, \tag{1}$$

where k is a constant called the *thermal diffusivity*. This is an example of a "partial differential equation" – a relation between the partial derivatives of the function u. For example, $u(x, t) = e^{-kt} \cos(x)$ is a solution of (1), because

$$\frac{\partial u}{\partial t} = -ke^{-kt} \cos(x)$$

while

$$\frac{\partial^2 u}{\partial x^2} = -e^{-kt} \cos(x).$$

So $\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}$. The partial derivatives can be checked by Maple as above!

Lab Problem B

Verify that each of the following functions satisfies the heat equation (1) and answer the questions connected with them:

1) $u_1(x, t) = ax + b$ (a, b constant). What is "special" about this type of solution? (What happens to the temperature at each x as t changes?)
2) $u_2(x, t) = x(x^2 + 6kt)$.
3) $u_3(x, t) = e^{-kt} \sin(x)$.
4) $$u_4(x, t) = \frac{1}{\sqrt{t}} e^{\frac{-x^2}{4kt}}$$

   Note that u_4 is not defined at $t = 0$. Plot $u_4(x, t)$ using the `plot3d` command with $k = 1$ for $x \in [-10, 10]$, and $t \in [.01, 10]$, and describe what is happening as t increases. (You may take as known that heat energy flows from areas of high temperature to areas of low temperature.) What do you think the graph of $\lim_{t \to 0^+} u_4(x, t)$ looks like? What does the slice of the graph $z = u_4(x, t)$ in the plane $x = 1$ look like? What does the shape of this slice mean physically? Explain.

Let $u(x, t)$ represent the vertical displacement at position x, and time t of a long thin string (like a piano or guitar string) extending from $x = 0$ to $x = \pi$. Assume the ends of the string are fixed, but the rest of it can vibrate. It can be shown that the displacement function satisfies the following relation, called the *wave equation*:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \tag{2}$$

where c is a constant. This is another example of a "partial differential equation" – a relation between the partial derivatives of the function u.

Lab Problem C

Verify that each of the following functions satisfies the wave equation (2) and answer the accompanying questions:

1) $u_m(x,t) = \sin(mx)\cos(mct)$, and $v_m(x,t) = \sin(mx)\sin(mct)$, where $m \geq 1$ is any integers.
2) Show that the following "linear combination" of $u_m(x,t)$ and $v_m(x,t)$ also satisfies the wave equation (2):

   $$u(x,t) = u_1(x,t)+4v_3(x,t)-3u_4(x,t) = \sin(x)\cos(ct)+4\sin(3x)\sin(3ct)-3\sin(4x)\cos(4ct).$$

   This combination might represent a particular motion of a vibrating string.
3) Take $c = 1$, and plot the linear combination from part 2 using `plot3d`. Explain the motion of the vibrating string by taking the vertical slices of the graph in planes defined by $t =$ constant.

Assignment

Individual write-ups due in class on Friday, October 15.