Mathematics 41, section 1 – Multivariable Calculus
Discussion 8 – Polar Double Integrals, Centers of Mass
November 18, 1999

Background

Here's a joke to try out on your friends:

Question: What is a polar bear? Answer: It's a rectangular bear after the change of coordinates

$$x = r\cos\theta, \quad y = r\sin\theta.$$

Yesterday, we saw that if $f(x, y)$ was any function, and R was a region in the plane given in polar coordinates by

$$R = \{(r, \theta) : a \le r \le b, \alpha \le \theta \le \beta\},$$

then the double integral

$$\int\int_R f\,dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\,r dr d\theta.$$

Today we will look at how polar coordinates can simplify the computations of things like centers of mass *when the geometry of the region R is well adapted to that coordinate system.* First, recall the *average value of f on R* is given by:

$$\frac{\int\int_R f(x,y)\,dA}{\text{area}(R)} = \frac{\int\int_R f(x,y)\,dA}{\int\int_R 1\,dA}$$

In particular, if $f = x$, or $f = y$ we would get

$$\overline{x} = \frac{\int\int_R x\,dA}{\int\int_R 1\,dA}$$
$$\overline{y} = \frac{\int\int_R y\,dA}{\int\int_R 1\,dA}$$

and the point $(\overline{x}, \overline{y})$ can be thought of as the *center of mass* of the region R (it would be the center of mass of a thin plate in the shape of R if the material of the plate had constant density).

Discussion Questions

A) Compute the center of mass of the region R in the first and second quadrants bounded by the positive x-axis, the line $y = -x$, and an arc of the circle of radius 2 centered at the origin. Suggestion: think polar! *Answer:*

$$(\overline{x}, \overline{y}) = \left(\frac{8\sqrt{2}}{9\pi}, \frac{8\sqrt{2}+16}{9\pi}\right) \doteq (.4001, .9660).$$

B) Suppose you had a thin plate in the shape of the region R from question A, but now the density of the material at each point was variable, given by the function $\rho(x, y) = x^2 + y^2$. What double integrals would compute total mass of the plate and its center of mass? (Think of $\overline{x}$ and $\overline{y}$ as "weighted averages," weighted by the density function.)

C) Next, suppose we wanted to determine the center of mass of an object that had thickness. For simplicity, suppose our object is a rectangular solid $S = [a, b] \times [c, d] \times [e, f]$ in $\mathbf{R}^3$, but suppose that the object has varying density $\rho(x, y, z)$ at each point $(x, y, z) \in S$. How could you approximate the total mass and the coordinates of the center of mass?

Assignment

Group write-ups due Monday, November 22.