Mathematics 41, section 1 – Multivariable Calculus
Discussion 5 – The Vibrating Drumhead
October 29, 1999

Background

The *wave equation* describes motions of a vibrating square drumhead. If u is the vertical displacement of the point $(x, y) \in [-\pi, \pi] \times [-\pi, \pi]$ at time t, then

$$\frac{\partial^2 u}{\partial t^2} = \frac{1}{2}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \tag{1}$$

Yesterday in class we began discussing a particular solution

$$u(x, y, t) = \sin(x)\sin(y)\sin(t)$$

of (1) describing one particular fundamental mode of vibration.

We did not mention this in class, but note that at all points on the boundary of the square $[-\pi, \pi] \times [-\pi, \pi]$, $u(x, y, t) = 0$ for all t. In other words, the edges of the drumhead are fixed at zero vertical displacement – just like a real drumhead!

If t is frozen at one particular time, $t = t_0$, then for $u(x, y, t_0)$,

$$\frac{\partial u}{\partial x}(x, y, t_0) = \cos(x)\sin(y)\sin(t_0)$$
$$\frac{\partial u}{\partial y}(x, y, t_0) = \sin(x)\cos(y)\sin(t_0)$$

So if $\sin(t_0) \neq 0$, the critical points of $u(x, y, t_0)$ in interior of the square are: $(x, y) = (0, 0), \left(\frac{\pi}{2}, \frac{\pi}{2}\right), \left(\frac{\pi}{2}, \frac{-\pi}{2}\right), \left(\frac{-\pi}{2}, \frac{\pi}{2}\right), \left(\frac{-\pi}{2}, \frac{-\pi}{2}\right)$ and 8 additional points on the boundary edges that we will not be concerned with.

Discussion Questions

A) Use the Second Derivative Test to classify each of the 5 critical points in the interior as a local maximum, a local minimum, or a saddle point. Your answer should depend on what t_0 is – how does that work? By considering how the character of the critical points changes with t, describe the motion of the drumhead in words for $t \in [0, 4\pi]$.

B) Show that $u(x, y, t) = \sin(2x)\sin(y)\sin((\sqrt{5/2})t)$ also solves the wave equation (1) and the "boundary conditions" $u(x, -\pi, t) = u(x, \pi, t) = u(-\pi, y, t) = u(\pi, y, t) = 0$ (these are the equations that say the points on the edges of the drumhead have zero vertical displacement for all times). Find the critical points for this u, determine how their type (local maximum, local minimum, or saddle point) varies with t, and describe this motion, comparing it with your answer from part A. What differences are there? Apart from the difference in the numbers of critical points, what is the effect of the factor of $\sqrt{5/2}$ in the $\sin((\sqrt{5/2})t)$?

C) Another question we could ask: In the solution from part A, at $t = \pi/2$, what are the highest and lowest points on the drumhead with (x, y) on the circle $(x-1)^2+y^2 = 4$? In other words, what are the maximum and minimum values of $u(x, y, \pi/2)$ if we also impose the *constraint* that $x^2 + y^2 = 1$? Here is some potentially useful information for questions like this: a plot of the gradient vector field for $f(x, y) = u(x, y, \pi/2) = \sin(x)\sin(y)$, overlaid with the circle and the contours of f. Estimate the points on the circle where the largest and smallest values of f are attained. What is happening at those points if you look at the circle and the contours of f?

Assignment

Group writeups due Wednesday, November 3.