Mathematics 41, section 1 – Multivariable Calculus
Discussion 3 – Limits for $f(x, y)$
October 13, 1999

## Background

As is implied by their name, the *partial derivatives* of a function $f(x, y)$ are not the "whole story" to understanding how $f$ is changing at $(x_0, y_0)$ in its domain. We also want to develop something like a "total derivative" of $f$ at $(x_0, y_0)$, and to do this we need to understand *limits* of functions $f(x, y)$ as $(x, y) \to (x_0, y_0)$. In other words, we want to understand what it should mean to say

$$\lim_{(x,y)\to(x_0,y_0)} f(x, y) = L$$

for some real number $L$.

## Discussion Questions

A first observation is that we can let $(x, y) \to (x_0, y_0)$ in many ways in $\mathbf{R}^2$. For instance, given a direction vector $\vec{v} = (v_1, v_2)$, we could let $(x, y) = (x_0 + tv_1, y_0 + tv_2)$ (on the line through $(x_0, y_0)$ in the direction of $\vec{v}$), and take an ordinary limit as $t \to 0$:

(1) $$\lim_{t\to 0} f(x_0 + tv_1, y_0 + tv_2)$$

A) For instance, what happens if you let $f(x, y) = x^2 - 2xy + 4y^2$, $(x_0, y_0) = (1, 2)$, and take a limit as in (1) for several different $\vec{v}$? Does the choice of $\vec{v}$ seem to matter? Explain. Does it make to sense to say $\lim_{(x,y)\to(1,2)} f(x, y) = L$ for some $L$? If so, which $L$ "works"? If not, why not?

B) Now, suppose

$$g(x, y) = \begin{cases} \frac{xy}{x^2+y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

1) What happens if you proceed as in A for $(x_0, y_0) = (1, 1)$ and take several different $\vec{v}$? Does the choice of $\vec{v}$ seem to matter? Explain.
2) Does it make to sense to say $\lim_{(x,y)\to(1,1)} g(x, y) = L$ for some $L$? If so, which $L$ "works"? If not, why not?
3) Now repeat part 1 for $(x_0, y_0) = (0, 0)$. Is this case different? Why or why not. You may want to refer to the graph and contour plot of $g(x, y)$ on the accompanying Maple worksheet.
4) Does it make to sense to say $\lim_{(x,y)\to(0,0)} g(x, y) = L$ for some $L$? If so, which $L$ "works"? If not, why not?

C) Now, let

$$h(x,y) = \begin{cases} \frac{y^2-x^4}{y^2+x^4} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

1) What happens if you proceed as in A for $(x_0, y_0) = (0,0)$ and take several different $\vec{v}$? Does the choice of $\vec{v}$ seem to matter? Explain. Refer to the graph and contour plot of $h(x,y)$ on the accompanying Maple worksheet.
2) Before you decide whether $\lim_{(x,y)\to(0,0)} h(x,y)$ should exist, also consider letting $(x,y) \to (0,0)$ by letting $t \to 0$ on the following two parabolas: $(x,y) = (t,t^2)$ and $(x,y) = (t,2t^2)$. and computing

$$\lim_{t\to 0} h(t,t^2)$$

and

$$\lim_{t\to 0} h(t,2t^2).$$

3) Does it make to sense to say $\lim_{(x,y)\to(0,0)} h(x,y) = L$ for some $L$? If so, which $L$ "works"? If not, why not?

D) What should it mean to say $\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L$?

**Assignment**

Group write-ups due Monday, October 18.