Mathematics 241, section 1 – Multivariable Calculus
Solutions for Exam III – November 26, 2013

I. Given that

$$
\begin{aligned}
f(x,y) &= \frac{y}{x^2+y^2+4}\\
f_x(x,y) &= \frac{-2xy}{x^2+y^2+4)^2}\\
f_y(x,y) &= \frac{x^2-y^2+4}{(x^2+y^2+4)^2}\\
f_{xx}(x,y) &= \frac{2y(3x^2-y^2-4)}{(x^2+y^2+4)^3}\\
f_{xy}(x,y) &= \frac{-2x(x^2-3y^2+4)}{(x^2+y^2+4)^3}\\
f_{yy}(x,y) &= \frac{-2y(3x^2-y^2+12)}{(x^2+y^2+4)^3},
\end{aligned}
$$

find all critical points of f and apply the Second Derivative Test to each to determine the types of the critical points.

Solution: The critical points are the solutions of $f_x = f_y = 0$. From $f_x = 0$, we get either $x = 0$ or $y = 0$. There are no solutions of the equation $f_y = 0$ with $y = 0$, but if $x = 0$, we get $y = \pm 2$. So the critical points are $(0,2)$ and $(0,-2)$. At $(0,2)$, $f_{xx}(0,2)f_{yy}(0,2) - (f_{xy}(0,2))^2 = (-1/16)(-1/16) - 0^2 = 1/256 > 0$ and $f_{xx}(0,2) = -1/16 < 0$. Therefore, f has a *local maximum* at $(0,2)$. At $(0,-2)$, $f_{xx}(0,-2)f_{yy}(0,-2) - (f_{xy}(0,-2))^2 = (1/16)(1/16) - 0^2 = 1/256 > 0$. and $f_{xx}(0,-2) = 1/16 > 0$. Therefore f has a *local minimum* at $(0,-2)$.

II.

A) (20) Find the maximum and minimum values of

$$f(x,y) = xy + 2y$$

on the closed and bounded set

$$S = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 \le 9\}.$$

Solution: We have

$$
\begin{aligned}
f_x &= y\\
f_y &= x+2
\end{aligned}
$$

So f has one critical point at $(-2, 0) \in S$ with $f(-2, 0) = 0$. (This is a saddle point for f). We use Lagrange Multipliers next. With the constraint equation $x^2 + y^2 = 9$, the Lagrange equations are

$$\begin{aligned} y &= 2\lambda x \\ x + 2 &= 2\lambda y \\ x^2 + y^2 - 9 &= 0. \end{aligned}$$

If we multiply the first equation by y and the second by x, then we can equate the right sides to obtain an equation in which λ has been eliminated: $y^2 = x(x + 2) = x^2 + 2x$. We can then use this equation to substitute for the y^2 and eliminate y in the constraint equation:

$$2x^2 + 2x - 9 = 0.$$

This equation can be solved by the quadratic formula:

$$x = \frac{-2 \pm \sqrt{4 + 72}}{4} \doteq -2.68, 1.68.$$

Then from the constraint equation we get two y-values for each of these x-values, so there are four solutions, approximately

$$(-2.68, \pm 1.35), (1.68, \pm 2.49)$$

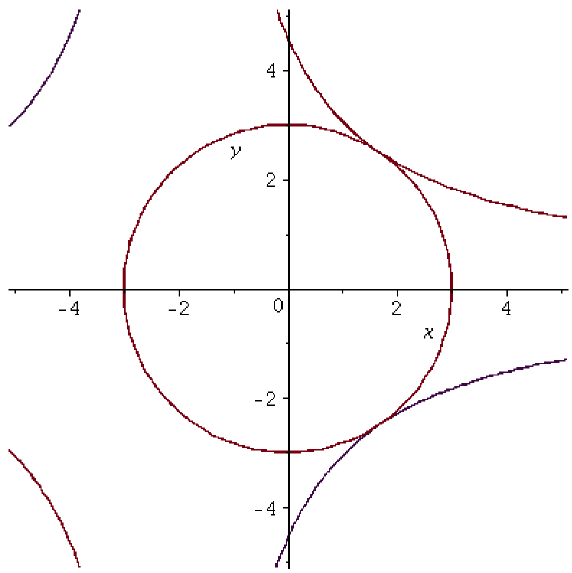
We have

$$\begin{aligned} f(-2, 0) &= 0 \\ f(-2.68, 1.35) &\doteq -.92 \\ f(-2.68, -1.35) &\doteq .92 \\ f(1.68, 2.49) &\doteq -9.2 \\ f(1.68, -2.49) &\doteq 9.2 \end{aligned}$$

So putting everything together, we see that the maximum value of f on S is about 9.2 and the minimum value of f on S is about -9.2.

B) (5) Give a sketch of the contours of f for the maximum and minimum values you found in part A.

Solution: The function is $f(x, y) = (x + 2)y$. Hence the contours of f for values not equal to zero are *rectangular hyperbolas* with asymptotes equal to the lines $x = -2$ and $y = 0$.



The graphs shows two hyperbolas, *each tangent to the boundary circle of S at the points where the maximum and minimum values are attained.* The relevant portions of the branches of the hyperbolas are in the first and fourth quadrants of the plane. (Recall the geometry of Lagrange Multipliers we saw in the Lab portion of Problem Set 7.)

III. All parts of this problem refer to the region

$$R = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 4, y \geq 0, y \leq 2x\}.$$

and the solid $S \subset \mathbb{R}^3$ with upper boundary $z = x^2$ and lower boundary the xy-plane *and the "shadow" of S in the xy-plane is R.*

A) (10) Sketch the region R in the plane.

Solution: This is the region inside a circle of radius 2, above the x-axis, and below the line $y = 2x$: Note: the region lies entirely in the first quadrant.

B) (10) Set up iterated integral(s) of the form $\int\int f(x, y)\, dy\, dx$ to compute the volume of S. Do not compute the value.

Solution: Since the upper boundary of R changes when $y = 2x$ crosses the circle, we need the sum of two interated integrals. The line $y = 2x$ crosses $x^2 + y^2 = 4$ when $5x^2 = 4$ so $x = \pm 2/\sqrt{5}$. Since our region is entirely in the first quadrant, it's

$$\int_0^{2/\sqrt{5}} \int_0^{2x} x^2\, dy\, dx + \int_{2/\sqrt{5}}^{2} \int_0^{\sqrt{4-x^2}} x^2\, dy\, dx.$$

C) (10) Now set up iterated integral(s) of the form $\int\int f(x,y)\ dx\ dy$ to compute the volume of S. Do not compute the value.

Solution: This one is slightly tricky because you have to notice that, even though the integral does not need to be split, the region R does not extend all the way up to $y=2$. The range of y-values is determined like this: The line $y=2x$ crosses $x^2+y^2=4$ when $5x^2=4$ so $x=2/\sqrt{5}$ and $y=4/\sqrt{5}$. The integral is:

$$\int_0^{4/\sqrt{5}} \int_{y/2}^{\sqrt{4-y^2}} x^2\ dx\ dy.$$

D) (10) Finally, set up a polar coordinate integral that will compute the volume of S. Do not compute the value.

Solution: With polar coordinates, you do not need to split into two integrals, and the limits of integration are all constant. The part of the line $y=2x$ in the first quadrant is the ray $\theta=\arctan(2)$. With the "extra r" from the dA in polar coordinates:

$$\int_0^{\arctan(2)} \int_0^2 r^3\cos^2\theta\ dr\ d\theta.$$

E) (10) Finally, compute the volume of S using any one of your answers from parts B,C,D (your choice).

Solution: Don't know about you, but the first two look pretty bad to me! Let's evaluate the last one:

$$\begin{aligned}
\int_0^{\arctan(2)} \int_0^2 r^3\cos^2\theta\ dr\ d\theta &= \int_0^2 r^3\ dr \cdot \int_0^{\arctan(2)} \cos^2\theta\ d\theta \\
&= \left.\frac{r^4}{4}\right|_0^2 \cdot \left.\left(\frac{\theta}{2}+\frac{\sin(2\theta)}{4}\right)\right|_0^{\arctan(2)} \\
&= 4\cdot\left(\frac{1}{2}\arctan(2)+\frac{1}{5}\right).
\end{aligned}$$

Extra Credit (10) Evaluate the improper double integral

$$\int\int_{\mathbb{R}^2} e^{1-x^2-y^2}\ dA = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{1-x^2-y^2}\ dy\ dx.$$

We can evaluate this by switching to polar coordinates! The integral over all of $\mathbb{R}^2$ is

$$\int_0^{2\pi}\int_0^{\infty} e^{1-r^2} r\ dr\ d\theta$$

This equals

$$2\pi \cdot \lim_{B\to\infty} -\frac{1}{2}e^{1-r^2}\Big|_0^B = 2\pi \cdot \lim_{B\to\infty} \left(\frac{e}{2} - \frac{1}{2}e^{1-B^2}\right) = \pi e.$$

Wow!