Mathematics 241, section 1 – Multivariable Calculus
Midterm Exam 2 Solutions
November 1, 2013

I. All parts of this problem refer to the vector field

$$\mathbf{F}(x,y) = (x^2 - 2x, xy - y).$$

A. (10) Find all critical points of $\mathbf{F}(x,y)$.

Solution: The critical points are the solutions of the simultaneous system

$$\begin{aligned} x^2 - 2x &= x(x-2) = 0 \\ xy - y &= (x-1)y = 0 \end{aligned}$$

The solutions are $(0,0)$ and $(2,0)$.

B. (5) There are two vector fields plotted on the back of this sheet. Say which one shows $\mathbf{F}(x,y)$ and use that plot to classify each of the critical points as a source, sink, saddle, or center.

Solution: This is Vector Field 2 in the plots (the critical points of Vector Field 1 are not at the right locations). From the plot, $(0,0)$ is a *sink* and $(2,0)$ is a *source.*

C. (20) Show that $\alpha(t) = (0, 4e^{-t})$ and $\beta(t) = \left(\frac{2}{1+e^{2t}}, 0\right)$ are both flow lines of F. What are $\lim_{t\to\infty} \alpha(t)$ and $\lim_{t\to\infty} \beta(t)$?

Solution: For $\alpha(t)$ we compute $\alpha'(t) = (0, -4e^{-t})$. On the other hand $(\mathbf{F}\circ\alpha)(t) = (0^2 - 0, 0 \cdot 4e^{-t} - 4e^{-t}) = (0, -4e^{-t})$. Therefore $\alpha(t)$ is a flow line of $\mathbf{F}$. For $\beta(t)$, similarly, we have

$$\beta'(t) = \left(\frac{-4e^{2t}}{(1+e^{2t})^2}, 0\right)$$

On the other hand,

$$\begin{aligned} (\mathbf{F}\circ\beta)(t) &= \left(\left(\frac{2}{1+e^{2t}}\right)^2 - \frac{4}{1+e^{2t}}, 0\right) \\ &= \left(\frac{4 - 4(1+e^{2t})}{(1+e^{2t})^2}, 0\right) \\ &= \left(\frac{-4e^{2t}}{(1+e^{2t})^2}, 0\right). \end{aligned}$$

D. (5) Is there a scalar-valued function $f(x,y)$ such that $\mathbf{F}(x,y) = \nabla f(x,y)$? Why or why not?

Solution: No, there is not. The reason is that we would need to have $\frac{\partial f}{\partial y} = xy - y$, so $f(x,y) = \frac{xy^2}{2} - \frac{y^2}{2} + g(x)$, for some function $g(x)$. But then $\frac{\partial f}{\partial x} = \frac{y^2}{2} + g'(x)$. There is no y^2 in the first component of $\mathbf{F}$, so this is not possible.

II. In the neighborhood of Eagle Pass, the landscape has elevation in feet above sea level given by $f(x,y) = \frac{x^2}{4} - y^2 + 1000$.

A. (10) Sketch the contours of $f(x,y)$ for $c = 999, 1000, 1001$ on the same set of axes.

Solution: The contours for $c = 999$ and $c = 1001$ are hyperbolas, the contour for $c = 1000$ is the union of the two lines $y = \pm\frac{x}{2}$. (Those lines are the asymptote lines of the hyperbolas.)

B. (10) Compute the directional derivative $D_u f(2,1)$ for a general unit vector.

Solution: We have $\nabla f(x,y) = \left(\frac{x}{2}, -2y\right)$, so $\nabla f(2,1) = (1,-2)$. The directional derivative $D_u f(2,1) = \nabla f(2,1) \cdot u = u_1 - 2u_2$.

C. (5) In the direction of which unit vector u should you walk from the point with $(x,y) = (2,1)$ in order to decrease your elevation at the fastest rate?

Solution: The unit vector in the direction of $-\nabla f(2,1)$, so

$$u = \left(\frac{-1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right)$$

III. All parts of this problem refer to the function

$$f(x,y) = \frac{x^3 - 3xy^2}{x^2 + y^2} \text{ if } (x,y) \neq (0,0) \text{ and } f(0,0) = 0.$$

A. (15) Find the tangent plane to $z = f(x,y)$ at $(1,1,f(1,1))$. We have, at $(x,y) \neq (0,0)$:

$$\begin{aligned}
\frac{\partial f}{\partial x} &= \frac{(x^2+y^2)(3x^2-3y^2) - (x^3-3xy^2)(2x)}{(x^2+y^2)^2} \\
&= \frac{x^4 + 6x^2y^2 - 3y^4}{(x^2+y^2)^2} \\
\therefore \frac{\partial f}{\partial x}(1,1) &= \frac{4}{4} = 1.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\frac{\partial f}{\partial y} &= \frac{(x^2+y^2)(-6xy) - (x^3-3xy^2)(2y)}{(x^2+y^2)^2} \\
&= \frac{-8x^3y}{(x^2+y^2)^2} \\
\therefore \frac{\partial f}{\partial x}(1,1) &= \frac{-8}{4} = -2.
\end{aligned}$$

The tangent plane is $z = -1 + (1)(x-1) + (-2)(y-1)$, or after simplifying: $z = x - 2y$.

B. (10) Does $\frac{\partial f}{\partial x}(0,0)$ exist? If so, find it; if not say why not.

Solution: By the limit definition,

$$\begin{aligned}
\frac{\partial f}{\partial x}(0,0) &= \lim_{h\to 0} \frac{f(0+h,0) - f(0,0)}{h} \\
&= \lim_{h\to 0} \frac{h^3/h^2 - 0}{h} \\
&= \lim_{h\to 0} 1 \\
&= 1.
\end{aligned}$$

So the answer is yes.

IV. (10) Can the curve $\alpha(t) = (2\cos(t), \sin(t))$ for $t \in (0, \infty)$ be a flow line of the vector field $\nabla f(x, y)$ for a differentiable function f? Why or why not?

Solution: The answer is NO. Notice that $\alpha(t)$ is an ellipse with the usual counterclockwise parametrization and $\alpha(0) = \alpha(2\pi) = (2, 0)$. If this was a flow line for the gradient vector field for some $f(x, y)$, then as we know, $f(x, y)$ would be steadily *increasing with* t as we move along the flow line. However that is not possible since $f(\alpha(0)) = f(\alpha(2\pi))$.

Extra Credit (10) Refer to the function in question III. Let m be arbitrary and compute $\lim_{t\to 0} f(t, mt)$ (the limit of the value of f along the line through the origin in the direction of the vector $(1, m)$). Is $\lim_{t\to 0}(f \circ \alpha)(t) = 0$ for *every* differentiable curve $\alpha(t)$ with $\alpha(0) = (0, 0)$? Explain.

Solution: We have

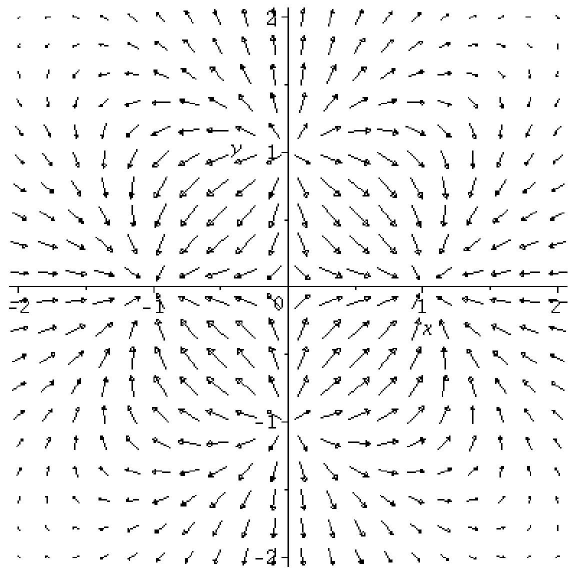
$$f(t, mt) = \frac{t^3(1 - 3m^2)}{t^2(1 + m^2)} = \frac{t(1 - 3m^2)}{(1 + m^2)}$$

Hence $\lim_{t\to 0} f(t, mt) = 0$ for all m. It will be true that $\lim_{t\to 0}(f \circ \alpha)(t) = 0$ here because

$$\frac{x^3 - 3xy^2}{x^2 + y^2} = x \cdot \frac{x^2 - 3y^2}{x^2 + y^2}$$

the second factor takes only values between 1 and -3, while the $x \to 0$ if we are moving along any curve $\alpha(t)$ with $\alpha(0) = 0$. By the squeeze theorem, the limit must exist and equal zero.

Vector Field 1:



Vector Field 2: