Mathematics 241, section 1 – Multivariable Calculus
Solutions for Exam 1
October 1, 2010

I. All parts of this question refer to the three points $P = (1, 0, 1)$, $Q = (1, 2, -2)$, and $R = (1, -1, 1)$ in $\mathbf{R}^3$.

A) (15) Find the equation of the plane containing P, Q, R.

Solution: The vector from P to Q is $Q - P = (0, 2, -3)$ and the vector from P to R is $R - P = (0, -1, 0)$. Hence $N = (Q - P) \times (R - P) = (-3, 0, 0)$ is a normal vector for the plane we want. Using $\frac{-1}{3}N$ and the point P, equation is

$$x = 1.$$

(Note that this could also be derived by observation without any calculations!)

B) (15) Give a parametrization of the line segment from R to Q (in that direction), including the proper range of t-values. Which t-value gives the midpoint?

Solution: The direction vector we want is $Q - R = (0, 3, -3)$. The line segment is

$$(1, -1, 1) + t(0, 3, -3) = (1, -1 + 3t, 1 - 3t) \quad \text{where } 0 \le t \le 1.$$

The midpoint is obtained with $t = 1/2$. (The line segment is parametrized with a constant speed curve, so over the interval $[0, 1]$, the midpoint is obtained for $t = 1/2$.)

C) (10) Compute the angle between the vectors $v = Q - P$ and $w = R - P$.

Solution: $Q - P = (0, 2, -3)$ and $R - P = (0, -1, 0)$ from part A. The angle satisfies

$$\cos(\theta) = \frac{(0, 2, -3) \cdot (0, -1, 0)}{\sqrt{13} \cdot 1} = \frac{-2}{\sqrt{13}}.$$

The angle is

$$\theta = \cos^{-1}\left(\frac{-2}{\sqrt{13}}\right) \doteq 2.16 \text{ radians}.$$

II. II. Let $\alpha(t) = (u(t), v(t))$ and $\beta(t) = (w(t), z(t))$ be parametric curves in $\mathbf{R^2}$.

A) (10) Show that if $\alpha(t)$ and $\beta(t)$ are differentiable, then

$$\frac{d}{dt}(\alpha(t) \cdot \beta(t)) = \alpha'(t) \cdot \beta(t) + \alpha(t) \cdot \beta'(t),$$

(where $\cdot$ denotes the dot product).
Solution: We have

$$\alpha(t) \cdot \beta(t) = u(t)w(t) + v(t)z(t).$$

So by the product and sum rules for derivatives:

$$\begin{aligned}\frac{d}{dt}(\alpha(t)\cdot\beta(t)) &= u'(t)w(t)+u(t)w'(t)+v'(t)z(t)+v(t)z'(t)\\ &= (u'(t)w(t)+v'(t)z(t))+(u(t)w'(t)+v(t)z'(t))\\ &= \alpha'(t)\cdot\beta(t)+\alpha(t)\cdot\beta(t).\end{aligned}$$

This is what we wanted to show.

B) (10) Let $\beta(t)$ be a parametric curve in $\mathbf{R^2}$. Suppose that $\beta(t)\cdot\beta(t)=1$ (dot product) for all t. Show that the vector $\beta'(t)$ is orthogonal to the vector $\beta(t)$ for all t.
Solution: Using part A and the commutativity of the dot product, we see

$$0=\frac{d}{dt}(\beta(t)\cdot\beta(t))=2\beta(t)\cdot\beta'(t).$$

Therefore $\beta(t)\cdot\beta'(t)=0$, which shows that they are orthogonal.

III. All parts of this question refer to

$$\alpha(t)=(\cos(2t)\cos(t),\cos(2t)\sin(t))$$

called a *4-leaved rose curve.*

A) (15) Starting from $t=0$, what is the first t with $\alpha(t)=(0,0)$? At how many different t is $\alpha(t)=(0,0)$ in the range $0\le t<\pi$?

Solution: Starting from $t=0$, the first t with $\alpha(t)=(0,0)$ is $t=\pi/4$. There are two t in the range $0\le t<\pi$ where $\alpha(t)=(0,0)$: $t=\pi/4,3\pi/4$. These all come from zeroes of the function $\cos(2t)$.

B) (15) Find a parametrization of the tangent line to the rose curve at $t=\pi/4$.

Solution: We have

$$\alpha(\pi/4)=(\cos(\pi/2)\cos(\pi/4),\cos(\pi/2)\sin(\pi/4))=(0,0).$$

Then by the product rule in each component,

$$\alpha'(t)=(-2\sin(2t)\cos(t)-\cos(2t)\sin(t),-2\sin(2t)\sin(t)+\cos(2t)\cos(t)).$$

Hence

$$\alpha'(\pi/4)=(-\sqrt{2},-\sqrt{2}).$$

The tangent line is

$$\{(-s\sqrt{2},-s\sqrt{2})\mid s\in\mathbf{R}\}.$$

C) (10)Give a parametrization of a circle that will fit completely inside the "leaf" of $\alpha(t)$ in quadrants I and IV. (Any circle that does that is OK.)

Solution: If we place the center at $(1/2, 0)$, then to fit inside the "leaf" in quadrants I and IV, the radius must be small, something like $1/10$. Something like

$$\beta(t) = (1/2 + \cos(t)/10, \sin(t)/10)$$

is a reasonable answer.

Extra Credit (10) Identify and sketch the slices of the surface

$$\{(x, y, z) \in \mathbf{R}^3 \mid x^2 - y^2 + z^2 = 1\}$$

in the planes $x = 1$ and $x = 2$.

Solution: The slice in the plane $x = 1$ is given by $-y^2 + z^2 = 0$. This is the union of two lines: $x = 1, y = z$, and $x = 1, y = -z$. The slice in the plane $x = 2$ is given by $-y^2 + z^2 = -3$, or $\frac{y^2}{3} - \frac{z^2}{3} = 1$. This is a hyperbola opening in the direction of the positive and negative y-axes.