Mathematics 241 – Multivariable Calculus
Solutions for Final Examination – December 14, 2013

I. In this problem, $P = (1, 0, 1)$, $Q = (-2, 3, 2)$, and $R = (1, 2, 0)$.

A) Find the equation of the plane containing the points P, Q, R in $\mathbf{R}^3$.

Solution: The displacement vector from P to Q is $\mathbf{v} = Q - P = (-3, 3, 1)$ and the vector from P to R is $\mathbf{w} = R - P = (0, 2, -1)$. For the plane we can take $N = (-3, 3, 1) \times (0, 2, -1) = (-5, -3, -6)$. Then the equation of the plane is $0 = N \cdot (x - 1, y - 0, z - 1) = -5x + 5 - 3y - 6z + 6$, or $5x + 3y + 6z = 11$.

B) At what point does the line containing P, Q meet the xy-plane?

Solution: The line is $(1, 0, 1) + (-3, 3, 1)t = (1 - 3t, 3t, 1 + t)$. This meets the xy-plane when $z = 1 + t = 0$, so $t = -1$. The point of intersection is $(4, -3, 0)$.

Comment: Several people substituted the parametrization into the equation of the plane from part A. *Read questions more carefully, and don't just memorize a sample problem from a practice exam!*

C) If $\mathbf{v}$ is the displacement vector from P to Q and $\mathbf{w}$ is the displacement vector from P to R, at what angle do $\mathbf{v}, \mathbf{w}$ meet?

Solution: The angle θ satisfies $\cos\theta = \frac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{v}\|\|\mathbf{w}\|} = \frac{5}{\sqrt{19}\sqrt{5}}$. So

$$\theta = \cos^{-1}\left(\sqrt{5/19}\right) \doteq 1.032 \text{ radians.}$$

II. All parts of this problem refer to the parametric curve

$$\alpha(t) = \left(\frac{\cos(t)}{\sin^2(t) + 1}, \frac{\cos(t)\sin(t)}{1 + \sin^2(t)}\right)$$

defined for all $t \in [0, 2\pi]$, called a *lemniscate*.

A) Is $\alpha(t)$ a simple closed curve? (Hint: Thinking of $\alpha(t)$ as the position of a moving object as a function of time, are there different times $t \in [0, 2\pi)$ at which the object is at the location $(x, y) = (0, 0)$?)

Solution: We have $\alpha(0) = \left(\frac{1}{2}, 0\right) = \alpha(2\pi)$, so this is a closed curve. However, $\cos(t) = 0 = \cos(t)\sin(t)$ for $t = \frac{\pi}{2}$ and $t = \frac{3\pi}{2}$ in the given interval. Since there are two such t, the answer to the first question is *NO*.

B) What is the tangent vector to the curve at $t = \pi$?

Solution: By the quotient rule in each component,

$$\alpha'(t) = \frac{1}{(1 + \sin^2(t))^2}(-(1 + \sin^2(t))\sin(t) - 2\cos^2(t)\sin(t),$$
$$(1 + \sin^2(t))(\cos^2(t) - \sin^2(t)) - 2\cos^2(t)\sin^2(t))$$

When $t - \pi$, we get $\alpha'(\pi) = (0, 1)$

C) A thin wire has the shape of the portion of the curve α for $t \in [0, 1]$. What integral would you use to compute its arc length. (You do not need to evaluate!)

Solution: The arc length would be computed by

$$M = \int_0^1 ds = \int_0^1 \|\alpha'(t)\| dt$$

III. All parts of this problem refer to $f(x, y) = (x + 1)^2 + y^2$.

A) Sketch the level curves of $f(x, y)$ for the values $c = 1, 4, 9$.

Solution: The level curves of f are circles with center at $(-1, 0)$ the radii are $r = 1, 2, 3$ respectively.

B) At the point $(1, 2)$, in which direction is f increasing the fastest? Express your answer as a unit direction vector.

Solution: This is in the direction of the gradient vector $\nabla f(1, 2)$. The gradient vector is $\nabla f(x, y) = (2(x + 1), 2y)$ at a general point. So $\nabla f(1, 2) = (4, 4)$. The unit vector in this direction is $\frac{1}{4\sqrt{2}}(4, 4) = (\sqrt{2}/2, \sqrt{2}/2)$.

C) Find the points on the curve $g(x, y) = \frac{x^2}{4} + y^2 = 1$ at which $f(x, y)$ takes its largest and smallest values. What is true about the vectors ∇f and ∇g at your points?

Solution: Using the Lagrange multiplier method, we must solve

$$\begin{aligned} 2(x + 1) &= \lambda x/2 \\ 2y &= 2\lambda y \\ \frac{x^2}{4} + y^2 &= 1 \end{aligned}$$

From the second equation, $y = 0$ or $\lambda = 1$. If $y = 0$, the constraint equation gives $x = \pm 2$, so we obtain two points $(\pm 2, 0)$. If $\lambda = 1$, then from the first equation, $2(x + 1) = x/2$, so $x = -4/3$. Then from the constraint equation we get $y = \pm\sqrt{5}/3$. To determine which of these give maximum and minimum values, we substitute into $f(x, y)$:

$$\begin{aligned} f(2, 0) &= 9 \text{ (maximum)} \\ f(-2, 0) &= 1 \\ f(-4/3, \pm\sqrt{5}/3) &= 1/9 + 5/9 = 2/3 \text{ (minimum)} \end{aligned}$$

The points we found here are the points where the level curve of f passing through that point and the constraint curve are *tangent*.

IV. Let $f(x, y) = xe^{-2x^2 - y^2}$.

A) Find the equation of the tangent plane to the graph $z = f(x, y)$ at the point $(1, 1, e^{-3})$.

Solution: We must compute the partial derivatives to start:

$$f_x = (1 - 4x^2)e^{-2x^2-y^2}$$
$$f_y = -2xye^{-2x^2-y^2}.$$

At $(x, y) = (1, 1)$, $f_x(1, 1) = -3e^{-3}$, and $f_y(1, 1) = -2e^{-3}$, so the tangent plane is

$$z = e^{-3} - 3e^{-3}(x - 1) - 2e^{-3}(y - 1).$$

B) Find all the critical points of $f(x, y)$.

Solution: The critical points are the solutions of $f_x = 0$ and $f_y = 0$. Using the formulas for f_x, f_y from part A, we see that $f_x = 0$ when $x = \pm 1/2$ and $f_y = 0$ when $x = 0$ or $y = 0$ (Note: the exponential factor is *never zero*.) Hence the simultaneous solutions are the two points $(\pm 1/2, 0)$.

C) Use the second derivative test (Hessian criterion) to determine the type of each critical point you found in part B.

Solution: Now we need the second-order partial derivatives as well:

$$f_{xx} = (16x^3 - 12x)e^{-2x^2-y^2}$$
$$f_{xy} = (1 - 4x^2)(-2y)e^{-2x^2-y^2}$$
$$f_{yy} = -2x(1 - 2y^2)e^{-2x^2-y^2}$$

So at $(1/2, 0)$ the Hessian matrix is

$$\begin{pmatrix} -4e^{-1/2} & 0 \\ 0 & -e^{-1/2} \end{pmatrix}$$

The determinant is $4e^{-1} > 0$ and the upper left entry is < 0 so this is a *local maximum.* At $(-1/2, 0)$ the Hessian matrix is

$$\begin{pmatrix} 4e^{-1/2} & 0 \\ 0 & e^{-1/2} \end{pmatrix}$$

The determinant is $4e^{-1} > 0$ and the upper left entry is > 0 so this is a *local minimum.*

V. A region R in $\mathbf{R}^2$ is the set of points satisfying $x^2 + y^2 \geq 1$, $y \geq x$, $x \geq 0$, and $y \leq 4$.

A) Sketch the region R.

Solution: This is the region outside the unit circle with center $(0, 0)$, to the right of the y-axis, below the horizontal line $y = 4$, and above the line $y = x$.

B) Set up the limits of integration of iterated integral(s) to compute $\int\int_R f(x, y)\, dA$ integrating with respect to x first, then y.

Solution: The circle intersects the line $y = x$ at $(\sqrt{2}/2, \sqrt{2}/2)$. From there to the top of the circle at $y = 1$, the left boundary of the region is part of the circle. For $y > 1$, though, the left boundary is part of the y-axis so we have to split the integral at $y = 1$:

$$\int_{\sqrt{2}/2}^{1} \int_{\sqrt{1-y^2}}^{y} f(x, y)\ dx\ dy + \int_{1}^{4} \int_{0}^{y} f(x, y)\ dx\ dy.$$

C) Now reverse the order of the variables and set up iterated integral(s) to compute the same integral, but integrating with respect to y first, then x.

Solution: We also need to split the integral this way since the bottom boundary changes at $x = \sqrt{2}/2$. The region extends all the way to $x = 4$ on the right, where the line $y = 4$ intersects $y = x$:

$$\int_{0}^{\sqrt{2}/2} \int_{\sqrt{1-x^2}}^{4} f(x, y)\ dy\ dx + \int_{\sqrt{2}/2}^{4} \int_{x}^{4} f(x, y)\ dy\ dx.$$

VI. (20) The metal making up a solid half-cone in the shape of

$$H = \{(x, y, z) \in \mathbf{R}^3 \mid z^2 \geq x^2 + y^2, 0 \leq z \leq 1, y \geq 0\}$$

has density $\delta(x, y, z) = \sqrt{x^2 + y^2 + z^2}$ at all points. Determine its total mass. (Hint: a wise choice of coordinate system will simplify this one immensely!)

Solution: We will set up the triple integral to compute the mass using *spherical* coordinates, since the spherical equation of the boundary cone is $\phi = 1$. The restriction $y \geq 0$ says $0 \leq \theta \leq \pi$. The slightly tricky thing here is that this is a cone with a planar base, not the *"snow-cone"* region described by $0 \leq \rho \leq 1$, which has a rounded top from a part of the sphere $\rho = 1$. In spherical coordinates, the plane $z = 1$ is $\rho \cos \phi = 1$, so $\rho = \sec \phi$ on the outer boundary. The density is just $\delta = \rho$. So the total mass is

$$\begin{aligned}
M &= \int_{0}^{\pi} \int_{0}^{\pi/4} \int_{0}^{\sec \phi} \rho^3 \sin \phi d\rho\ d\phi\ d\theta \\
&= \int_{0}^{\pi} \int_{0}^{\pi/4} \int_{0}^{\sec \phi} \rho^3 \sin \phi d\rho\ d\phi\ d\theta \\
&= \int_{0}^{\pi} \int_{0}^{\pi/4} \left. \frac{\rho^4}{4} \right|_{0}^{\sec \phi} \sin \phi\ d\phi\ d\theta \\
&= \int_{0}^{\pi} \int_{0}^{\pi/4} \frac{\sin \phi}{4 \cos^4 \phi}\ d\phi\ d\theta \quad (u^{-4} du) \\
&= \pi \left. \frac{1}{12 \cos^3(\phi)} \right|_{0}^{\pi/4} \\
&= \frac{\pi}{12}(2\sqrt{2} - 1)
\end{aligned}$$

Comment: If you set this up in cylindrical coordinates instead, it would look like this: The density is $\delta = \sqrt{r^2+z^2}$ and the limits of integration would be $0 \le \theta \le \pi$, $0 \le r \le 1$ (the "shadow region"), and $r \le z \le 1$, since the top half of the cone $z^2 = x^2 + y^2$ is $z = r$. This forces the z-integral to come first, though, since the limit of integration depends on r:

$$\int_0^\pi \int_0^1 \int_r^1 \sqrt{r^2+z^2}\ r\ dz\ dr\ d\theta.$$

Note that you *do not* have the du for $u = r^2 + z^2$, integrating *with respect to* z. This is a (much) harder integral that would need to be done with a tangent substitution, then parts on the resulting $\sec^3$ form.

VII.

A) State Green's Theorem.

Solution: If C is a simple closed curve, positively oriented, D is the interior region enclosed by C, and $\mathbf{F}(x,y) = (F_1(x,y), F_2(x,y))$ is a C^1 vector field on D, then

$$\oint_C F \cdot T\ ds = \oint_C F_1 dx + F_2 dy = \int\int_D \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\ dA.$$

B) (10) Let $\mathbf{F}(x,y) = (x - y^2, x^2 + y)$. Verify that Green's Theorem holds for the region $D = \{(x,y) \in \mathbf{R}^2 : x^2 + y^2 \le 9\}$.

Solution: Using the standard parametrization $(x,y) = (3\cos(t), 3\sin(t))$ of the boundary circle of D,

$$\begin{aligned}\oint_{\partial D} F_1 dx + F_2 dy &= \int_0^{2\pi} (3\cos(t) - 9\sin^2(t))(-3\sin(t)) \\ &\quad + (9\cos^2(t) + 3\sin(t))(3\cos(t))\ dt \\ &= 27 \int_0^{2\pi} \sin^3(t) + \cos^3(t)\ dt \\ &= 27\left(-\frac{2}{3}\cos(t) - \frac{1}{3}\sin^2(t)\cos(t) + \frac{2}{3}\sin(t) + \frac{1}{3}\cos^2(t)\sin(t)\right)\Bigg|_0^{2\pi} \\ &= 0.\end{aligned}$$

The double integral over D is

$$\int\int_D (F_2)_x - (F_1)_y\ dA = \int\int_D 2x + 2y\ dA.$$

This can be evaluated in a number of ways. Switching to polar coordinates, for instance,

$$= \int_0^{2\pi} \int_0^3 2r^2(\cos\theta + \sin\theta)\ dr\ d\theta = 0$$

since both $\int_0^{2\pi} \cos\theta \; d\theta = 0$ and $\int_0^{2\pi} \sin\theta \; d\theta = 0$.

VIII. A function $f(x, y)$ is said to be *harmonic* on an open set U in $\mathbf{R}^2$ if it satisfies the equation

$$f_{xx} + f_{yy} = 0$$

at all points in U.

A) How does a nondegenerate critical point of a harmonic function fit into our classification? Is it a local maximum, local minimum, or a saddle point? Explain how you can tell from the second derivative test.

Solution: Every nondegenerate critical point of a harmonic function is a *saddle point* because the Hessian matrix is

$$D^2(f) = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{pmatrix} = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{xy} & -f_{xx} \end{pmatrix}.$$

The determinant here is $-(f_{xx})^2 - (f_{xy})^2 < 0$.

B) If f is harmonic, what is true about the line integral of the vector field

$$\mathbf{F}(x, y) = (-f_y, f_x)$$

around any simple closed curve in U?

Solution: Let D be the region bounded by the simple closed curve. By Green's theorem, the integral is equal to

$$\int\int_D (f_x)_x - (-f_y)_y \; dA = \int\int_D f_{xx} + f_{yy} \; dA = 0.$$

Extra Credit (10)

Suppose you follow a flow line of the vector field $-\nabla f$ for $f(x, y)$ in the xy-plane. As you traverse the flow line in the increasing t-direction, is the corresponding path on the graph $z = f(x, y)$ going uphill or downhill? Explain. What does the vector field $-\nabla f$ look like near a local maxmimum of f? near a local minimum of f?

Solution: You are always going *downhill* by the most direct route – recall $\nabla f(a, b)$ gives the direction in which f is increasing the fastest. The negative gradient vector field near a local maximum will have all arrows pointing away from the critical point (flow lines will diverge from the maximum). Near a local minimum, the negative gradient vector field will be pointing toward from the crirical point (flow lines will be converging toward the minimum).

Have a peaceful and joyous holiday season!