Mathematics 241 – Multivariable Calculus
Final Examination – December 14, 2013

Directions

Do all work in the blue exam booklet. Do not place anything you wish to have considered for credit on the exam sheet. There are 200 regular and 10 extra credit points, distributed as indicated next to the questions.

Notation: In the following, for simplicity, the subscript notation is used for partial derivatives: $f_x = \frac{\partial f}{\partial x}$, $f_y = \frac{\partial f}{\partial y}$, $f_{xx} = \frac{\partial^2 f}{\partial x^2}$, and so forth.

I. In this problem, $P = (1, 0, 1)$, $Q = (-2, 3, 2)$, and $R = (1, 2, 0)$.

A) (10) Find the equation of the plane containing the points P, Q, R in $\mathbf{R}^3$.
B) (10) At what point does the line containing P, Q meet the xy-plane?
C) (10) If $\mathbf{v}$ is the displacement vector from P to Q and $\mathbf{w}$ is the displacement vector from P to R, at what angle do $\mathbf{v}, \mathbf{w}$ meet?

II. All parts of this problem refer to the parametric curve

$$\alpha(t) = \left(\frac{\cos(t)}{\sin^2(t) + 1}, \frac{\cos(t)\sin(t)}{1 + \sin^2(t)}\right)$$

defined for all $t \in [0, 2\pi]$, called a *lemniscate*.

A) (5) Is $\alpha(t)$ a simple closed curve? (Hint: Thinking of $\alpha(t)$ as the position of a moving object as a function of time, are there different times $t \in [0, 2\pi)$ at which the object is at the location $(x, y) = (0, 0)$?)
B) (10) What is the tangent vector to the curve at $t = \pi$?
C) (5) A thin wire has the shape of the portion of the curve α for $t \in [0, 1]$. What integral would you use to compute its arc length? (You do not need to evaluate!)

III. All parts of this problem refer to $f(x, y) = (x + 1)^2 + y^2$.

A) (5) Sketch the level curves of $f(x, y)$ for the values $c = 1, 4, 9$.
B) (10) At the point $(1, 2)$, in which direction is f increasing the fastest? Express your answer as a unit direction vector.
C) (15) Find the points on the curve $g(x, y) = \frac{x^2}{4} + y^2 = 1$ at which $f(x, y)$ takes its largest and smallest values. What is true about the vectors ∇f and ∇g at your points?

IV. Let $f(x, y) = xe^{-2x^2 - y^2}$.

A) (10) Find the equation of the tangent plane to the graph $z = f(x, y)$ at the point $(1, 1, e^{-3})$.
B) (10) Find all the critical points of $f(x, y)$.
C) (20) Use the second derivative test (Hessian criterion) to determine the type of each critical point you found in part B. (If you have difficulty, partial credit will be given

for a correct statement of the second derivative test even if you cannot see how to apply it.)

V. A region R in $\mathbf{R}^2$ is the set of points

$$R = \{(x, y) \mid x^2 + y^2 \geq 1, y \geq x, x \geq 0, \text{ and } y \leq 4.\}$$

A) (10) Sketch the region R.
B) (10) Set up the limits of integration of iterated integral(s) to compute $\int\int_R f(x, y)\, dA$ integrating with respect to x first, then y.
C) (10) Now reverse the order of the variables and set up iterated integral(s) to compute the same integral, but integrating with respect to y first, then x.

Note: In parts B and C, you are just to set up integrals with a "dummy function" $f(x, y)$, there is no evaluation of integrals involved.

VI. (20) The metal making up a solid half-cone in the shape of

$$H = \{(x, y, z) \in \mathbf{R}^3 \mid z^2 \geq x^2 + y^2, 0 \leq z \leq 1, y \geq 0\}$$

has density $\delta(x, y, z) = \sqrt{x^2 + y^2 + z^2}$ at all points. Determine its total mass. (Hint: a wise choice of coordinate system will simplify this one immensely!)

VII.

A) (10) State Green's Theorem.
B) (10) Let $\mathbf{F}(x, y) = (x - y^2, x^2 + y)$. Verify that Green's Theorem holds for the region $D = \{(x, y) \in \mathbf{R}^2 : x^2 + y^2 \leq 9\}$.

VIII. A function $f(x, y)$ is said to be *harmonic* on an open set U in $\mathbf{R}^2$ if it satisfies the equation

$$f_{xx} + f_{yy} = 0$$

at all points in U.

A) (5) How does a nondegenerate critical point of a harmonic function fit into our classification? Is it a local maximum, local minimum, or a saddle point? Explain how you can tell from the second derivative test.
B) (5) If f is harmonic, what is true about the line integral of the vector field

$$\mathbf{F}(x, y) = (-f_y, f_x)$$

around any simple closed curve in U?

Extra Credit (10)

Suppose you follow a flow line of the vector field $-\nabla f$ for a differentiable function $f(x, y)$ in the xy-plane. As you traverse the flow line in the increasing t-direction, is the corresponding path on the graph $z = f(x, y)$ going uphill or downhill? Explain. What does the vector field $-\nabla f$ look like near a local maxmimum of f? near a local minimum of f?

Have a peaceful and joyous holiday season!