Mathematics 241, section 1 – Multivariable Calculus
Solutions for Exam 1
October 1, 2010

I. All parts of this question refer to the three points $P = (0, 0, 1)$, $Q = (1, 2, -4)$, and $R = (1, -1, 0)$ in $\mathbf{R}^3$.

A) (15) Find the equation of the plane containing P, Q, R.

Solution: The vector from P to Q is $Q - P = (1, 2, -5)$ and the vector from P to R is $R - P = (1, -1, -1)$. Hence $N = (Q - P) \times (R - P) = (-7, -4, -3)$ is a normal vector for the plane we want. Using N and the point P, equation is $(-7, -4, -3) \cdot (x - 0, y - 0, z - 1) = 0$, or

$$7x + 4y + 3z = 3.$$

B) (10) Give a parametrization of the line segment from R to Q (in that direction), including the proper range of t-values.

Solution: The direction vector we want is $Q - R = (0, 3, -4)$. The line segment is

$$(1, -1, 0) + t(0, 3, -4) = (1, -1 + 3t, -4t) \quad \text{where } 0 \le t \le 1.$$

C) (10) Which t-value gives the midpoint of the line segment from part B (the point equidistant from R and Q) in your parametrization?

Solution: Since the line segment is traversed at constant speed in this parametrization between $t = 0$ and $t = 1$, the midpoint will be reached at $t = \frac{1}{2}$. The midpoint is $M = (1, \frac{1}{2}, -2)$. It can be checked that $\|M - Q\| = \|M - R\| = \frac{5}{2}$.

D) (10) Compute the angle between the vectors $\vec{PQ}$ and $\vec{PR}$.

Solution: $Q - P = (1, 2, -5)$ and $R - P = (1, -1, -1)$. The angle satisfies

$$\cos(\theta) = \frac{(1, 2, -5) \cdot (1, -1, -1)}{\sqrt{30}\sqrt{3}} = \frac{4}{3\sqrt{10}} = \frac{4\sqrt{10}}{30}.$$

The angle is

$$\theta = \cos^{-1}\left(\frac{4\sqrt{10}}{30}\right).$$

II. Let $\mathbf{u}, \mathbf{v}, \mathbf{w}$ be vectors in $\mathbf{R}^3$ with tails at the origin.

A) (10) Show that

$$\mathbf{w} \times (\mathbf{u} + \mathbf{v}) = \mathbf{w} \times \mathbf{u} + \mathbf{w} \times \mathbf{v}.$$

Solution: Let $\mathbf{u} = (u_1, u_2, u_3)$, $\mathbf{v} = (v_1, v_2, v_3)$, and $\mathbf{w} = (w_1, w_2, w_3)$. Then

$$\begin{aligned}\mathbf{w} \times (\mathbf{u} + \mathbf{v}) &= (w_1, w_2, w_3) \times (u_1 + v_1, u_2 + v_2, u_3 + v_3) \\ &= (w_2(u_3 + v_3) - w_3(u_2 + v_2), \\ &\quad\ w_3(u_1 + v_1) - w_1(u_3 + v_3), w_1(u_2 + v_2) - w_2(u_1 + v_1)) \\ &= (w_2u_3 - w_3u_2, w_3u_1 - w_1u_3, w_1u_2 - w_2u_1) \\ &\quad\ + (w_2v_3 - w_3v_2, w_3v_1 - w_1v_3, w_1v_2 - w_2v_1) \\ &= \mathbf{w} \times \mathbf{u} + \mathbf{w} \times \mathbf{v}.\end{aligned}$$

B) (5) Show that if $\mathbf{w}$ is in the plane spanned by $\mathbf{u}$ and $\mathbf{v}$, then $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = 0$.

Solution: If $\mathbf{w}$ is in the plane spanned by $\mathbf{u}$ and $\mathbf{v}$, then there are scalars s, t such that $\mathbf{w} = s\mathbf{u} + t\mathbf{v}$. Then by another vector identity,

$$(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = (\mathbf{u} \times \mathbf{v}) \cdot (s\mathbf{u} + t\mathbf{v}) = s(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{u} + t(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{v}.$$

Since $\mathbf{u} \times \mathbf{v}$ is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$, both dot products here are zero. Hence $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = 0$.

III. All parts of this question refer to

$$\alpha(t) = (\cos(3t)\cos(t), \cos(3t)\sin(t))$$

called a *3-leaved rose curve.*

A) (15) Starting from $t = 0$, what is the first t with $\alpha(t) = (0, 0)$? At how many different t is $\alpha(t) = (0, 0)$ in the range $0 \leq t < \pi$?

Solution: Starting from $t = 0$, the first t with $\alpha(t) = (0, 0)$ is $t = \pi/6$. There are three t in the range $0 \leq t < \pi$ where $\alpha(t) = (0, 0)$: $t = \pi/6, \pi/2, 5\pi/6$. These all come from zeroes of the function $\cos(3t)$.

B) (15) Find a parametrization of the tangent line to the rose curve at $t = \pi/3$.

Solution: We have

$$\alpha(\pi/3) = (\cos(\pi)\cos(\pi/3), \cos(\pi)\sin(\pi/3)) = \left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right).$$

Then by the product rule in each component,

$$\alpha'(t) = (-3\sin(3t)\cos(t) - \cos(3t)\sin(t), -3\sin(3t)\sin(t) + \cos(3t)\cos(t)).$$

Hence

$$\alpha'(\pi/3) = \left(0 + \frac{\sqrt{3}}{2}, 0 - \frac{1}{2}\right) = \left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right).$$

The tangent line is

$$\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right) + s\left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right), s \in \mathbf{R}.$$

C) (10) Give a parametrization of a circle surrounding the "leaf" of $\alpha(t)$ in quadrants I and IV, *but not completely enclosing either of the other leaves.* (Any circle that does that is OK.)

Solution: If we place the center at $(1, 0)$, then to enclose the "leaf" in quadrants I and IV, but not the other two "leaves," we can take a radius greater than or equal to 1, but strictly less than the distance from $(1, 0)$ to the point $\alpha(\pi/3)$, which is $\sqrt{3}$. Note that $r = 1$ is in that range. Something like

$$\beta(t) = (1 + \cos(t), \sin(t))$$

is a reasonable answer.

Extra Credit (10) Let $\beta(t)$ be a parametric curve in $\mathbf{R}^3$. Suppose that $\beta(t) \cdot \beta(t) = 1$ (dot product) for all t. Show that $\beta'(t)$ is orthogonal to $\beta(t)$ for all t.

Solution: By the product formula for derivatives, it follows that

$$(\beta(t) \cdot \gamma(t))' = \beta'(t)\gamma(t) + \beta(t) \cdot \gamma'(t)$$

for any vector valued functions β and γ. If $\gamma = \beta$ and $\beta(t) \cdot \beta(t)$ is *constant*, then this shows

$$2\beta(t) \cdot \beta'(t) = 0$$

for all t. This shows that $\beta(t)$ and $\beta'(t)$ are orthogonal for all t.