Mathematics 241, section 1 – Multivariable Calculus
Solutions for Final Examination – December 14, 2010

I. In this problem,

$$P = (1, -1, 3),\ \ Q = (0, 3, 1),\ \text{ and } R = (-4, 1, 2).$$

A) (10) Find the equation of the plane containing the points P, Q, R in $\mathbf{R}^3$.

Solution: A normal vector is

$$\begin{aligned} N &= ((-4, 1, 2) - (1, -1, 3)) \times ((0, 3, 1) - (1, -1, 3)) \\ &= (-5, 2, -1) \times (-1, 4, -2) = (0, -9, -18) \end{aligned}$$

Since any scalar multiple of N is also normal to the plane, we can also use $N = (0, 1, 2)$, and then the equation becomes

$$(0, 1, 2) \cdot (x - 1, y + 1, z - 3) = 0$$

or

$$y + 2z - 5 = 0.$$

B) (10) At what point does the line containing P, Q meet the xy-plane?

Solution: The line containing P and Q can be parametrized as

$$\alpha(t) = (1, -1, 3) + t(-1, 4, -2) = (1 - t, -1 + 4t, 3 - 2t).$$

This meets the xy plane when $z = 0$ so $3 - 2t = 0$, or $t = 3/2$. The corresponding point is

$$\alpha(3/2) = (-1/2, 5, 0).$$

1. C) (5) If $\mathbf{v}$ is the vector from P to Q and $\mathbf{w}$ is the vector from P to R, at what angle do $\mathbf{v}, \mathbf{w}$ meet?

   Solution: The angle is θ satisfying

   $$\cos\theta = \frac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{v}\|\|\mathbf{w}\|} = \frac{15}{\sqrt{30}\sqrt{21}} = \frac{5}{\sqrt{70}}.$$

   This says

   $$\theta = \cos^{-1}\left(\frac{5}{\sqrt{70}}\right).$$

II. Suppose you follow a flow line of the vector field ∇f for $f(x, y)$ in the xy-plane.

A) (5) As you traverse the flow line in the increasing t-direction, is the corresponding path on the graph $z = f(x, y)$ going uphill or downhill? Explain.

Solution: Uphill – the gradient vector always points in the direction of maximum rate of increase.

B) (5) What does the vector field ∇f look like near a local maximum of f? Near a local minimum of f? Make rough sketches by hand to illustrate.

Solution: Near a local maximum, all the arrows from the vector field ∇f will be pointing "in" – the local maximum is a sink of the vector field. Near a local minimum, all the arrows will be pointing "out" – the local minimum is a source of the vector field.

III. All parts of this problem refer to $f(x, y) = (x - 1)^2 - y^2$.

A) (5) Sketch the contours of $f(x, y)$ for the values $c = -1, 0, 1$.

Solution: The contours for $c = \pm 1$ are hyperbolas with asymptotes along the lines $y = x - 1$ and $y = -x + 1$. The $c = -1$ contour opens up and down, and the $c = 1$ contour opens left and right. The $c = 0$ contour is the union of the two asymptotes.

B) (10) At the point $(1, 2)$, in which direction is f increasing the fastest? Express your answer as a unit direction vector.

Solution: The gradient vector $\nabla f(1, 2)$ points in this direction. $\nabla f(x, y) = (2(x - 1), -2y)$ so we have $\nabla f(1, 2) = (0, -4)$. The unit vector in this direction is $u = (0, -1)$.

IV. (20) After an ill-fated "three hour tour" goes awry, you are stranded on a island at the point with coordinates $(1, 1)$. Fortunately, you have a radio transmitter with you. Unfortunately, it has a limited range – its signal can only be received at distances less than or equal to $2/3$ from its position. You know that there is a Coast Guard patrol boat that makes a circuit of the path $x^2 + y^2 = 4$ every day, and they always carry a radio receiver and listen for transmissions. Will the patrol boat ever get within $2/3$ of your position and receive your signal? (Note: Minimizing the distance from a point (a, b) is the same as minimizing the function $f(x, y) = \|(x, y) - (a, b)\|^2$.)

Solution: One solution of this problem uses the method of Lagrange multipliers to find the point on the path of the patrol boat closest to $(1, 1)$. If the distance from the closest point to $(1, 1)$ is less than $2/3$, then the signal will be heard and you will be rescued. Using the hint, we want to minimize

$$f(x, y) = \|(x, y) - (1, 1)\|^2 = (x - 1)^2 + (y - 1)^2.$$

The constraint curve is $g(x,y) = x^2 + y^2 - 4 = 0$. The Lagrange equations are

$$\begin{aligned} 2(x-1) &= 2\lambda x \\ 2(y-1) &= 2\lambda y \\ x^2 + y^2 - 4 &= 0. \end{aligned}$$

We can eliminate λ between the first two equations by the usual method (multiply the first by y and the second by x, then equate the left sides), yielding

$$y(x-1) = x(y-1) \Rightarrow y = x.$$

In the constraint equation, this gives two points $(\sqrt{2}, \sqrt{2})$ and $(-\sqrt{2}, -\sqrt{2})$. The values of f at these points are

$$\begin{aligned} f(\sqrt{2}, \sqrt{2}) &= 2(\sqrt{2}-1)^2 = 6 - 4\sqrt{2} \\ f(-\sqrt{2}, -\sqrt{2}) &= 2(-\sqrt{2}-1)^2 = 6 + 4\sqrt{2}. \end{aligned}$$

Since $6 - 4\sqrt{2} \doteq .343 < (2/3)^2 \doteq .444$, your radio signal *will be* received by the patrol boat and you will be rescued!

Notes:

1. That the closest point to $(1,1)$ on the circle $x^2 + y^2 = 4$ is the point $(\sqrt{2}, \sqrt{2})$ can also be seen by elementary geometry. I gave full credit for solutions that showed this without using Lagrange multipliers.

2. The problem has a number of other correct solutions as well. Another way that many of you thought of is to find whether the circle $(x-1)^2 + (y-1)^2 = 4/9$ (the boundary of the region in which the signal from your receiver can be heard) intersects the path of the patrol boat $x^2 + y^2 = 4$. This can be determined by elementary algebra – the two circles intersect at approximately $(1.122, 1.655)$ and $(1.655, 1.122)$, so there is an arc on the path of the patrol boat consisting of points from which the signal can be heard.

V. Let $f(x,y) = ye^{-x^2-2y^2}$.

A) (10) Find the equation of the tangent plane to the graph $z = f(x,y)$ at the point $(1, 1, e^{-3})$.

Solution: We have $f_x = -2xye^{-x^2-2y^2}$ and $f_y = e^{-x^2-2y^2}(1 - 4y^2)$. So the tangent plane is

$$\begin{aligned} z &= f(1,1) + f_x(1,1)(x-1) + f_y(1,1)(y-1) \\ &= e^{-3} - 2e^{-3}(x-1) - 3e^{-3}(y-1). \end{aligned}$$

B) (10) Find all the critical points of $f(x,y)$.

Solution: We have f_x, f_y from part A. Setting those equal to zero, we obtain from $f_x = 0$ that $x = 0$ or $y = 0$. However $f_y = 0$ implies that $y = \pm 1/2$. So there are only two critical points $(0, \pm 1/2)$.

C) (20) Use the Second Derivative Test to determine the type of each critical point you found in part B.

Solution: We continue to compute the second-order partial derivatives.

$$\begin{aligned} f_{xx} &= y(-2+4x^2)e^{-x^2-2y^2} \\ f_{xy} &= x(-2+8y^2)e^{-x^2-2y^2} \\ f_{yy} &= (-12y+16y^3)e^{-x^2-2y^2} \end{aligned}$$

At $(0, 1/2)$,

$$\begin{aligned} A &= f_{xx}(0,1/2) = -e^{-1/2} \\ B &= f_{xy}(0,1/2) = 0 \\ C &= f_{yy}(0,1/2) = -4e^{-1/2}. \end{aligned}$$

Hence $AC - B^2 > 0$ and $A < 0$, so f has a local maximum at $(0, 1/2)$.

Similarly, at $(0, -1/2)$,

$$\begin{aligned} A &= f_{xx}(0,-1/2) = +e^{-1/2} \\ B &= f_{xy}(0,-1/2) = 0 \\ C &= f_{yy}(0,-1/2) = +4e^{-1/2}. \end{aligned}$$

Hence $AC - B^2 > 0$ and $A > 0$, so f has a local minimum at $(0, -1/2)$.

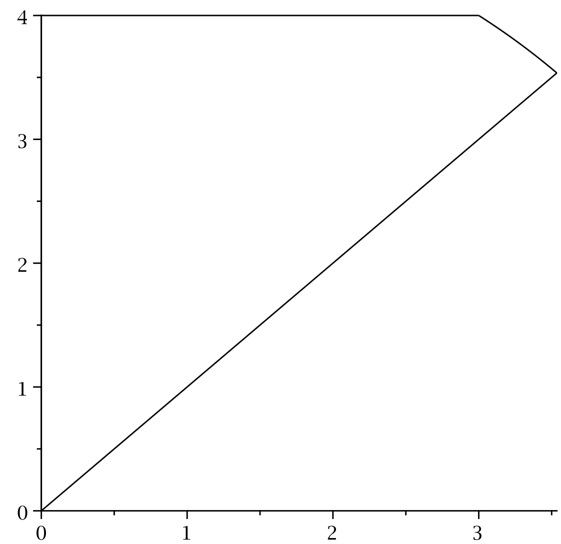
VI. The region R in $\mathbf{R}^2$ is the set of points

$$R = \{(x,y) \in \mathbf{R}^2 \mid x^2 + y^2 \leq 25, y \geq x, x \geq 0, \text{ and } y \leq 4\}$$

and let $f(x,y)$ be some continuous function defined on R.

A) (5) Sketch R.

Solution: This is the region inside the circle of radius 5 centered at $(0,0)$, above the line $y = x$, to the right of the y-axis, but below the line $y = 4$:



The line $y = 4$ intersects the circle at $(3, 4)$, and the line $y = x$ intersects at $(5\sqrt{2}/2, 5\sqrt{2}/2)$. Note that this has $y = 5\sqrt{2}/2 \doteq 3.54 < 4$. So the boundary of the region we want includes a small arc of the circle $x^2 + y^2 = 25$ between the points $(3, 4)$ and $(5\sqrt{2}/2, 5\sqrt{2}/2)$.

B) (10) Set up the limits of integration of iterated integral(s) to compute $\int\int_R f(x, y)\ dA$ integrating with respect to x first, then y.

Solution: This way, we must split the interval of y-values at $y = 5\sqrt{2}/2$, where the line $y = x$ crosses the circle:

$$\int_0^{5\sqrt{2}/2} \int_0^y f(x, y)\ dx\ dy + \int_{5\sqrt{2}/2}^4 \int_0^{\sqrt{25-y^2}} f(x, y)\ dx\ dy.$$

C) (10) Now reverse the order of the variables and set up iterated integral(s) to compute the same integral, but integrating with respect to y first, then x.

Solution: This one also must be split at $x = 3$ where the line $y = 4$ crosses the circle.

$$\int_0^3 \int_x^4 f(x, y)\ dy\ dx + \int_3^{5\sqrt{2}/2} \int_x^{\sqrt{25-x^2}} f(x, y)\ dy\ dx.$$

VII. (20) The metal making up a thin plate with the shape of the region in $\mathbf{R}^2$ with $x^2 + y^2 \leq 4$ has density $\delta(x, y) = 6 + y$ at all points. Determine the coordinates of its center of mass.

Solution: Since the plate has the shape of a circular disk, we will set up all the integrals in polar coordinates. First the total mass is

$$\begin{aligned} M = \int\int_R 6 + y \; dA &= \int_0^{2\pi} \int_0^2 (6 + r\sin\theta) r \; dr \; d\theta \\ &= \int_0^{2\pi} 12 + \frac{8}{3}\sin\theta \; d\theta \\ &= 24\pi. \end{aligned}$$

We can see $\overline{x} = 0$ since the plate and the density function are symmetric under $x \mapsto -x$. Finally

$$\begin{aligned} \overline{y} &= \frac{1}{24\pi} \int_0^{2\pi} \int_0^2 r\sin\theta(6 + r\sin\theta) r \; dr \; d\theta \\ &= \frac{1}{24\pi} \int_0^{2\pi} 16\sin\theta + 4\sin^2\theta \; d\theta \\ &= \frac{4\pi}{24\pi} \\ &= \frac{1}{6}. \end{aligned}$$

VIII. Consider the following triple integral:

$$\int_{-3}^{3} \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \int_{-\sqrt{9-x^2-y^2}}^{\sqrt{9-x^2-y^2}} 1 + x \; dz \; dy \; dx$$

A) (5) Describe the solid over which you are integrating here.

Solution: It is the solid sphere of radius 3 centered at $(0, 0, 0)$ in $\mathbf{R}^3$.

B) (20) Using any convenient coordinate system, evaluate this integral.

Solution: Spherical coordinates are probably the best choice:

$$\begin{aligned} &\int_0^{2\pi} \int_0^{\pi} \int_0^3 (1 + \rho\sin\phi\cos\theta)\rho^2 \sin\phi \; d\rho \; d\phi \; d\theta \\ &= \int_0^{2\pi} \int_0^{\pi} 9\sin\phi + \frac{81}{4}\sin^2\phi\cos\theta \; d\phi \; d\theta \\ &= \int_0^{2\pi} 18 + \frac{81\pi}{8}\cos\theta \; d\theta \\ &= 36\pi. \end{aligned}$$

This can also be done relatively easily in cylindrical coordinates, of course:

$$\int_0^{2\pi} \int_0^3 \int_{-\sqrt{9-r^2}}^{\sqrt{9-r^2}} (1 + r\cos\theta) r \; dz \; dr \; d\theta.$$

IX.

A) (10) State Green's Theorem.

Solution: If α is a differentiable simple, closed, positively oriented curve enclosing a region R and $F(x, y) = (u(x, y), v(x, y))$ is a vector field whose component functions u, v have continuous first order partial derivatives on the region R, then

$$\int_\alpha F \cdot T \; ds = \int\int_R v_x - u_y \; dA.$$

B) (10) Let $\mathbf{F}(x, y) = (x - y^2, x^2 + y)$. Verify that Green's Theorem holds for the region $D = \{(x, y) \in \mathbf{R}^2 : x^2 + y^2 \leq 9\}$ by computing both sides of the equation in the theorem and showing that they are equal.

Solution: The line integral can be computed using $\alpha(t) = (3\cos(t), 3\sin(t))$. This gives:

$$\begin{aligned} \int_\alpha F \cdot T \; ds &= \int_0^{2\pi} (3\cos(t) - 9\sin^2(t), 9\cos^2(t) + 3\sin(t)) \cdot (-3\sin(t), 3\cos(t)) \; dt \\ &= \int_0^{2\pi} 27\sin^3 t + 27\cos^3 t \; dt \\ &= 0. \end{aligned}$$

The double integral can be computed using polar coordinates:

$$\begin{aligned} \int\int_R 2x + 2y \; dA &= \int_0^{2\pi} \int_0^3 2r^2(\cos\theta + \sin\theta) \; dr \; d\theta \\ &= \int_0^{2\pi} 18(\cos\theta + \sin\theta) \; d\theta \\ &= 0. \end{aligned}$$

Extra Credit

A function $f(x, y)$ is said to be *harmonic* on an open set U in $\mathbf{R}^2$ if it satisfies the equation

$$f_{xx} + f_{yy} = 0$$

at all points in U. (Here f_x, f_y, f_{xx}, f_{yy} are the partial derivatives of the harmonic function f with respect to the indicated variables.)

A) (5) How does a nondegenerate critical point of a harmonic function fit into our classification? Is it a local maximum, local minimum, or a saddle point? Explain how you can tell from the Second Derivative Test.

Solution: Since $f_{yy} = -f_{xx}$ at all critical points,

$$f_{xx}f_{yy} - (f_{xy})^2 = -(f_{xx})^2 - (f_{xy})^2 < 0$$

at every nondegenerate critical point. They are all saddle points(!)

B) (5) If f is harmonic, what is true about the *total flux* of the vector field

$$\mathbf{F}(x, y) = \nabla f(x, y)$$

across any simple closed curve in U? Explain.

Solution: The divergence of $\nabla f = (f_x, f_y)$ is

$$(f_x)_x + (f_y)_y = f_{xx} + f_{yy} = 0.$$

Hence

$$\int_\alpha F \cdot N \, ds = \int\int_R f_{xx} + f_{yy} \, dA = 0$$

by Green's Theorem.