Mathematics 241, section 1 – Multivariable Calculus
Solutions for Midterm Exam 2
November 5, 2010

I. All parts of this problem refer to the vector field

$$\mathbf{F}(x, y) = (x^2 - 2x, y^2 + 2y).$$

A. (5) Find all critical points of $\mathbf{F}(x, y)$.

Solution: The critical points are the solutions of $x^2 - 2x = 0$ and $y^2 + 2y = 0$, so $x = 0, 2$ and $y = 0, -2$. This gives four critical points all together: $(0, 0)$, $(2, 0)$, $(0, -2)$ and $(2, -2)$.

B. (10) There are two vector fields plotted on the back of this sheet. Say which one shows $\mathbf{F}(x, y)$ and use that plot to classify each of the critical points as a source, sink, saddle, or center.

Solution: The correct plot is Vector Field Two, since Vector Field One apparently has only two critical points. Consulting the plot, we see that $(0, 0)$ and $(2, -2)$ are saddle points, $(2, 0)$ is a source, and $(0, -2)$ is a sink.

C. (10) Show that $\alpha(t) = \left(\frac{2}{1-2e^{2t}}, \frac{-2}{1-3e^{-2t}}\right)$ is a flow line of $\mathbf{F}$.

Solution: We must show that $\alpha'(t) = \mathbf{F}(\alpha(t))$. The left side is

$$\alpha'(t) = \left(\frac{8e^{2t}}{(1 - 2e^{2t})^2}, \frac{12e^{-2t}}{(1 - 3e^{-2t})^2}\right).$$

The right side is

$$\begin{aligned} \mathbf{F}(\alpha(t)) &= \left(\left(\frac{2}{1 - 2e^{2t}}\right)^2 - 2\left(\frac{2}{1 - 2e^{2t}}\right), \left(\frac{-2}{1 - 3e^{-2t}}\right)^2 + 2\left(\frac{-2}{1 - 3e^{-2t}}\right)\right) \\ &= \left(\frac{4 - 4(1 - 2e^{2t})}{(1 - 2e^{2t})^2}, \frac{4 - 4(1 - 3e^{-2t})}{(1 - 3e^{-2t})^2}\right) \\ &= \left(\frac{8e^{2t}}{(1 - 2e^{2t})^2}, \frac{12e^{-2t}}{(1 - 3e^{-2t})^2}\right). \end{aligned}$$

This shows what we wanted.

D. (10) Show that any $f(x, y) = \frac{x^3}{3} - x^2 + \frac{y^3}{3} + y^2 + c$ satisfies $\nabla f = \mathbf{F}$. What are the critical points of such an f? Classify each of them as a local maximum, local minimum, or saddle point. (Don't "start over" here; use previous parts of this question as appropriate.)

Solutions: For any such f, we have $\frac{\partial f}{\partial x} = x^2 - 2x$ and $\frac{\partial f}{\partial y} = y^2 + 2y$. So $\nabla f(x, y) = (x^2 - 2x, y^2 + 2y) = \mathbf{F}(x, y)$. By properties of gradient vector fields, we know that $f(x, y)$ has a local minimum at $(2, 0)$, a local maximum at $(0, -2)$, and saddle points at $(0, 0)$ and $(2, -2)$.

II. All parts of this problem refer to the function

$$f(x, y) = \frac{x^3 - 3xy^2}{x^2 + y^2} \text{ if } (x, y) \neq (0, 0) \text{ and } f(0, 0) = 0.$$

A. (10) Compute $\frac{\partial f}{\partial x}(x, y)$ and $\frac{\partial f}{\partial y}(x, y)$ for $(x, y) \neq (0, 0)$.

Solution: By the quotient rule

$$\begin{aligned} \frac{\partial f}{\partial x} &= \frac{(x^2+y^2)(3x^2-3y^2)-(x^3-3xy^2)(2x)}{(x^2+y^2)^2} \\ &= \frac{x^4+6x^2y^2-3y^4}{(x^2+y^2)^2}. \end{aligned}$$

and

$$\begin{aligned} \frac{\partial f}{\partial y} &= \frac{(x^2+y^2)(-6xy)-(x^3-3xy^2)(2y)}{(x^2+y^2)^2} \\ &= \frac{-8x^3y}{(x^2+y^2)^2}. \end{aligned}$$

B. (10) Do $\frac{\partial f}{\partial x}(0, 0)$ and $\frac{\partial f}{\partial y}(0, 0)$ exist? If so, find them; if not say why not.

Solution: The formulas from part A are not defined at $(0, 0)$. However, this just says we need to use the limit definition to see whether the partial derivatives exist. We find

$$\begin{aligned} \frac{\partial f}{\partial x}(0,0) &= \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h} \\ &= \lim_{h\to 0} \frac{h^3/h^2-0}{h} \\ &= 1 \end{aligned}$$

and

$$\begin{aligned} \frac{\partial f}{\partial y}(0,0) &= \lim_{h\to 0} \frac{f(0,h)-f(0,0)}{h} \\ &= \lim_{h\to 0} \frac{0/h^2-0}{h} \\ &= 0. \end{aligned}$$

C. (10) Let m be arbitrary and compute $\lim_{t\to 0} f(t, mt)$ (the limit of the value of f along the line through the origin in the direction of the vector $(1, m)$). Does this *prove* anything about the existence of the limit $\lim_{(x,y)\to(0,0)} f(x, y)$? Explain your answer using the definition of the statement $\lim_{(x,y)\to(0,0)} f(x, y) = L$.

Solution: We have

$$f(t, mt) = \frac{(1-3m^2)t^3}{(1+m^2)t^2} = \left(\frac{1-3m^2}{1+m^2}\right) t.$$

So as $t \to 0$, the limit is zero. This *does not show* that $\lim_{(x,y)\to(0,0)} f(x, y) = 0$, though. For that, we would have to show that for all $\varepsilon > 0$, there exists $r > 0$ such that $|f(x, y)| < \varepsilon$ for all (x, y) in the open ball $B_r(0, 0)$, except possibly $(0, 0)$.

D. (10 Extra Credit) Is $f(x, y)$ *differentiable* at $(0, 0)$? Explain your answer.

Solution: The answer is no. To show differentiability, we would need to show that

$$\lim_{(x,y)\to(0,0)} \frac{f(x,y)-\ell(x,y)}{\sqrt{x^2+y^2}} = 0,$$

where $\ell(x,y)$ is the linear approximation at $(0,0)$. Here from part B we know $\ell(x,y) = x$, so

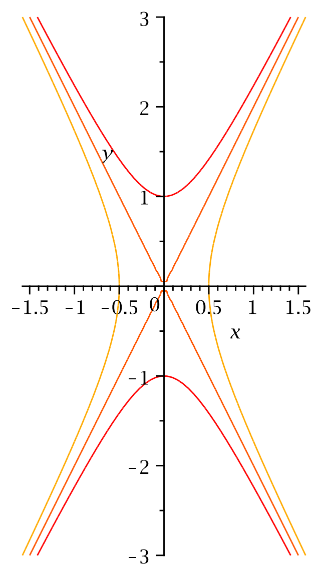
$$\frac{f(x,y)-\ell(x,y)}{\sqrt{x^2+y^2}} = \frac{-4xy^2}{(x^2+y^2)^{3/2}}.$$

If you let $(x,y) \to (0,0)$ along the x or y axes, this is going to 0. However, along the line $y = x$ this goes to the value $-4/2^{3/2} \neq 0$. Hence the required limit does not exist. Since it is not equal to zero, $f(x,y)$ is not differentiable.

III. All parts of this problem refer to $f(x,y) = 4x^2 - y^2$.

A. (15) Sketch the contours of $f(x,y)$ for $c = -1, 0, 1$ on the same set of axes.

Solution:



(These are hyperbolas; the $c = -1$ contour opens up and down; the $c = 0$ contour is the union of the two lines $y = \pm 2x$; the $c = 1$ contour opens left and right.)

B. (10) Consider

$$S = \{(x,y) \in \mathbb{R}^2 \mid -1 < f(x,y) < 0\}.$$

Is S open, closed, or neither? Why? (An intuitive explanation referring to the definition of open or closed is OK here.)

Solution: S is open since none of the boundary points of S (the points on the two contours for $c = -1$ and $c = 0$) are included in S. This means that for any $(x_0, y_0) \in S$, there exists a ball $B_r(x_0, y_0) \subset S$ for some $r > 0$.

C. (10) For which unit vectors u is the directional derivative $D_u f(1,2) = 0$?

Solution: The best way is to recognize that $f(1,2) = 4 - 4 = 0$. So $(1,2)$ is on the contour consisting of the two lines $y = \pm 2x$. We want u in the direction of the line with the plus sign to get the directional derivative equal to zero, so

$$u = \pm\left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right).$$