Mathematics 241 – Multivariable Calculus
Final Examination – December 11, 2007

I. In this problem, $P = (1, 0, 1)$, $Q = (-2, 3, 2)$, and $R = (1, 2, 0)$.

A) (10) Find the equation of the plane containing the points P, Q, R in $\mathbf{R}^3$.

Solution: The displacement vector from P to Q is $\mathbf{v} = Q - P = (-3, 3, 1)$ and the vector from P to R is $\mathbf{w} = R - P = (0, 2, -1)$. For the plane we can take $N = (-3, 3, 1) \times (0, 2, -1) = (-5, -3, -6)$. Then the equation of the plane is $0 = N \cdot (x - 1, y - 0, z - 1) = -5x + 5 - 3y - 6z + 6$, or $5x + 3y + 6z = 11$.

B) (10) At what point does the line containing P, Q meet the xy-plane?

Solution: The line is $(1, 0, 1) + (-3, 3, 1)t = (1 - 3t, 3t, 1 + t)$. This meets the xy-plane when $z = 1 + t = 0$, so $t = -1$. The point of intersection is $(4, -3, 0)$.

C) (5) If $\mathbf{v}$ is the displacement vector from P to Q and $\mathbf{w}$ is the displacement vector from P to R, at what angle do $\mathbf{v}, \mathbf{w}$ meet?

Solution: The angle θ satisfies $\cos\theta = \frac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{v}\|\|\mathbf{w}\|} = \frac{5}{\sqrt{19}\sqrt{5}}$. So

$$\theta = \cos^{-1}\left(\sqrt{5/19}\right) \doteq 1.032 \text{ radians.}$$

II.

A) (7.5) Let $F : \mathbf{R}^2_{u,v} \to \mathbf{R}^2_{x,y}$ be $F(u, v) = (x(u, v), y(u, v)) = (u^2 - v^2, 2uv)$ and let $g : \mathbf{R}^2_{x,y} \to \mathbf{R}$ be $g(x, y) = \sin(x)\cos(y)$. Find the derivative matrix $D(g \circ F)$ by direct substitution and differentiation.

Solution: The function $(g \circ F)(u, v) = \sin(u^2 - v^2)\cos(2uv)$, so computing partial derivatives by the product rule

$$\frac{\partial(g \circ F)}{\partial u} = 2u\cos(u^2 - v^2)\cos(2uv) - 2v\sin(u^2 - v^2)\sin(2uv)$$
$$\frac{\partial(g \circ F)}{\partial u} = -2v\cos(u^2 - v^2)\cos(2uv) - 2u\sin(u^2 - v^2)\sin(2uv)$$

and the derivative matrix is the 1×2 matrix with these entries.

B) (7.5) Compute $D(g \circ F)$ for the functions in part A using the Chain Rule and show you get the same result as in part A.

Solution: The Chain Rule says $D(g \circ F) = D(g)(F(u, v))D(F)(u, v)$ (matrix product). So we compute $D(g) = \begin{pmatrix} \cos(x)\cos(y) & -\sin(x)\sin(y) \end{pmatrix}$ and

$$D(F) = \begin{pmatrix} 2u & -2v \\ 2v & 2u \end{pmatrix}.$$

Hence the product matrix to be computed is

$$\begin{pmatrix} \cos(u^2 - v^2)\cos(2uv) & -\sin(u^2 - v^2)\sin(2uv) \end{pmatrix} \begin{pmatrix} 2u & -2v \\ 2v & 2u \end{pmatrix}$$

which gives a 1×2 matrix with first entry

$$2u\cos(u^2 - v^2)\cos(2uv) - 2v\sin(u^2 - v^2)\sin(2uv)$$

and second entry

$$-2v\cos(u^2 - v^2)\cos(2uv) - 2u\sin(u^2 - v^2)\sin(2uv)$$

as in the matrix $D(g \circ F)$.

C) (10) Now let $z = g(x, y)$ for a *general function* g – *not specifically* $g(x, y)$ *as above in part A*, and $x = u^2 - v^2$, $y = 2uv$ as in the definition of F above. Show that

$$\frac{1}{4(u^2 + v^2)}\left((z_u)^2 + (z_v)^2\right) = (z_x)^2 + (z_y)^2.$$

Solution: Now we use the expanded form of the Chain Rule: We have $z_u = z_x x_u + z_y y_u$ and $z_v = z_x x_v + z_y y_v$ so

$$\begin{aligned}
(z_u)^2 + (z_v)^2 &= (z_x x_u + z_y y_u)^2 + (z_x x_v + z_y y_v)^2 \\
&= (2uz_x + 2vz_y)^2 + (-2vz_x + 2uz_y)^2 \\
&= 4u^2(z_x)^2 + 8uvz_xz_y + 4v^2(z_y)^2 + 4v^2(z_x)^2 - 8uvz_xz_y + 4u^2(z_y)^2 \\
&= 4(u^2 + v^2)((z_x)^2 + (z_y)^2)
\end{aligned}$$

Dividing by $4(u^2 + v^2)$ yields the desired equation.

III. All parts of this problem refer to $f(x, y) = (x + 1)^2 + y^2$.

A) (5) Sketch the level curves of $f(x, y)$ for the values $c = 1, 4, 9$.

Solution: The level curves of f are circles with center at $(-1, 0)$ the radii are $r = 1, 2, 3$ respectively.

B) (10) At the point $(1, 2)$, in which direction is f increasing the fastest? Express your answer as a unit direction vector.

Solution: This is in the direction of the gradient vector $\nabla f(1, 2)$. The gradient vector is $\nabla f(x, y) = (2(x + 1), 2y)$ at a general point. So $\nabla f(1, 2) = (4, 4)$. The unit vector in this direction is $\frac{1}{4\sqrt{2}}(4, 4) = (\sqrt{2}/2, \sqrt{2}/2)$.

C) (15) Find the points on the curve $g(x, y) = \frac{x^2}{4} + y^2 = 1$ at which $f(x, y)$ takes its largest and smallest values. What is true about the vectors ∇f and ∇g at your points?

Solution: Using the Lagrange multiplier method, we must solve

$$\begin{aligned}
2(x + 1) &= \lambda x/2 \\
2y &= 2\lambda y \\
\frac{x^2}{4} + y^2 &= 1
\end{aligned}$$

From the second equation, $y = 0$ or $\lambda = 1$. If $y = 0$, the constraint equation gives $x = \pm 2$, so we obtain two points $(\pm 2, 0)$. If $\lambda = 1$, then from the first equation, $2(x+1) = x/2$, so $x = -4/3$. Then from the constraint equation we get $y = \pm\sqrt{5}/3$. To determine which of these give maximum and minimum values, we substitute into $f(x, y)$:

$$f(2,0) = 9 \text{ (maximum)}$$
$$f(-2,0) = 1$$
$$f(-4/3, \pm\sqrt{5}/3) = 1/9 + 5/9 = 2/3 \text{ (minimum)}$$

The points we found here are the points where the level curve of f passing through that point and the constraint curve are *tangent*.

IV. Let $f(x, y) = xe^{-2x^2-y^2}$.

A) (10) Find the equation of the tangent plane to the graph $z = f(x, y)$ at the point $(1, 1, e^{-3})$.

Solution: We must compute the partial derivatives to start:

$$f_x = (1 - 4x^2)e^{-2x^2-y^2}$$
$$f_y = -2xye^{-2x^2-y^2}.$$

At $(x, y) = (1, 1)$, $f_x(1, 1) = -3e^{-3}$, and $f_y(1, 1) = -2e^{-3}$, so the tangent plane is

$$z = e^{-3} - 3e^{-3}(x - 1) - 2e^{-3}(y - 1).$$

B) (10) Find all the critical points of $f(x, y)$.

Solution: The critical points are the solutions of $f_x = 0$ and $f_y = 0$. Using the formulas for f_x, f_y from part A, we see that $f_x = 0$ when $x = \pm 1/2$ and $f_y = 0$ when $x = 0$ or $y = 0$ (Note: the exponential factor is *never zero*.) Hence the simultaneous solutions are the two points $(\pm 1/2, 0)$.

C) (20) Use the second derivative test (Hessian criterion) to determine the type of each critical point you found in part B.

Solution: Now we need the second-order partial derivatives as well:

$$f_{xx} = (16x^3 - 12x)e^{-2x^2-y^2}$$
$$f_{xy} = (1 - 4x^2)(-2y)e^{-2x^2-y^2}$$
$$f_{yy} = -2x(1 - 2y^2)e^{-2x^2-y^2}$$

So at $(1/2, 0)$ the Hessian matrix is

$$D^2(f)(1/2, 0) = \begin{pmatrix} -4e^{-1/2} & 0 \\ 0 & -e^{-1/2} \end{pmatrix}$$

The determinant is $4e^{-1} > 0$ and the upper left entry is < 0 so this is a *local maximum.* At $(-1/2, 0)$ the Hessian matrix is

$$D^2(f)(-1/2,0) = \begin{pmatrix} 4e^{-1/2} & 0 \\ 0 & e^{-1/2} \end{pmatrix}$$

The determinant is $4e^{-1} > 0$ and the upper left entry is > 0 so this is a *local minimum.*

V. A region R in $\mathbf{R}^2$ is the set of points satisfying $x^2 + y^2 \geq 1$, $y \geq x$, $x \geq 0$, and $y \leq 4$.

A) (5) Sketch the region R.

Solution: This is the region outside the unit circle with center $(0, 0)$, to the right of the y-axis, below the horizontal line $y = 4$, and above the line $y = x$.

B) (10) Set up the limits of integration of iterated integral(s) to compute $\int\int_R f(x, y)\, dA$ integrating with respect to x first, then y.

Solution: The circle intersects the line $y = x$ at $(\sqrt{2}/2, \sqrt{2}/2)$. From there to the top of the circle at $y = 1$, the left boundary of the region is part of the circle. For $y > 1$, though, the left boundary is part of the y-axis so we have to split the integral at $y = 1$:

$$\int_{\sqrt{2}/2}^{1} \int_{\sqrt{1-y^2}}^{y} f(x,y)\; dx\; dy + \int_{1}^{4} \int_{0}^{y} f(x,y)\; dx\; dy.$$

C) (10) Now reverse the order of the variables and set up iterated integral(s) to compute the same integral, but integrating with respect to y first, then x.

Solution: We also need to split the integral this way since the bottom boundary changes at $x = \sqrt{2}/2$. The region extends all the way to $x = 4$ on the right, where the line $y = 4$ intersects $y = x$:

$$\int_{0}^{\sqrt{2}/2} \int_{\sqrt{1-x^2}}^{4} f(x,y)\; dy\; dx + \int_{\sqrt{2}/2}^{4} \int_{x}^{4} f(x,y)\; dy\; dx.$$

VI. (20) The metal making up a solid slug having the shape of the region in $\mathbf{R}^3$ with $x^2 + y^2 \leq 4$ and $-1 \leq z \leq 1$ has density $\delta(x, y, z) = 5 + x$ at all points. Determine the coordinates of its center of mass.

Solution: We will set up the triple integrals to compute the coordinates of the center of mass using *cylindrical* coordinates, since the slug is just a cylinder with axis along the z-axis. The total mass is

$$\begin{aligned}
M &= \int_0^{2\pi}\int_0^2\int_{-1}^1 (5+r\cos\theta) r dz\ dr\ d\theta \\
&= 2\int_0^{2\pi}\int_0^2 5r + r^2\cos\theta\ dr\ d\theta \\
&= 2\int_0^{2\pi} \frac{5}{2}r^2 + \frac{1}{3}r^3\cos\theta\Big|_0^2\ d\theta \\
&= 2\int_0^{2\pi} 10 + \frac{8}{3}\cos\theta\ d\theta \\
&= 20\theta + \frac{8}{2}\sin\theta\Big|_0^{2\pi} \\
&= 40\pi.
\end{aligned}$$

Since the density does not depend on y or z, by the symmetry of the cylinder, it can be seen that $\overline{y} = \overline{z} = 0$. To compute the x-coordinate,

$$\begin{aligned}
\overline{x} &= \frac{1}{40\pi}\int_0^{2\pi}\int_0^2\int_{-1}^1 r\cos\theta(5+r\cos\theta)r\ dz\ dr\ d\theta \\
&= \frac{1}{40\pi}\cdot 2\int_0^{2\pi}\int_0^2 5r^2\cos\theta + r^3\cos^2\theta\ dr\ d\theta \\
&= \frac{1}{20\pi}\int_0^{2\pi} \frac{5}{3}r^3\cos\theta + \frac{1}{4}r^4\cos^2\theta\Big|_0^2\ d\theta \\
&= \frac{1}{20\pi}\int_0^{2\pi} \frac{40}{3}\cos\theta + 4\cos^2\theta\ d\theta \\
&= \frac{1}{20\pi}\left(\frac{40}{3}\sin\theta + 2\theta + \cos(2\theta)\right)\Big|_0^{2\pi} \\
&= \frac{1}{20\pi}(0 + 4\pi + 1 - 0 - 0 - 1) \\
&= \frac{1}{5}.
\end{aligned}$$

The values of $\overline{y}$ and $\overline{z}$ can also be computed directly as above, replacing the $x = r\cos\theta$ by $y = r\sin\theta$ and z respectively.

VII.

A) (10) State Green's Theorem.

Solution: If D is a region in $\mathbf{R}^2$ bounded by a finite collection of simple closed curves, ∂D is the positively-oriented boundary of D, and $\mathbf{F}(x,y) = (M(x,y), N(x,y))$ is a C^1 vector field on D, then

$$\oint_{\partial D} Mdx + Ndy = \int\int_D N_x - M_y\ dA.$$

B) (10) Let $\mathbf{F}(x,y) = (x - y^2, x^2 + y)$. Verify that Green's Theorem holds for the region $D = \{(x,y) \in \mathbf{R}^2 : x^2 + y^2 \leq 9\}$.

Solution: Using the standard parametrization $(x,y) = (3\cos(t), 3\sin(t))$ of the boundary circle of D,

$$\begin{aligned}\oint_{\partial D} M dx + N dy &= \int_0^{2\pi} (3\cos(t) - 9\sin^2(t))(-3\sin(t)) \\ &\quad + (9\cos^2(t) + 3\sin(t))(3\cos(t))\ dt \\ &= 27 \int_0^{2\pi} \sin^3(t) + \cos^3(t)\ dt \\ &= 27\left(-\frac{2}{3}\cos(t) - \frac{1}{3}\sin^2(t)\cos(t) + \frac{2}{3}\sin(t) + \frac{1}{3}\cos^2(t)\sin(t)\right)\Bigg|_0^{2\pi} \\ &= 0.\end{aligned}$$

The double integral over D is

$$\iint_D N_x - M_y\ dA = \iint_D 2x + 2y\ dA.$$

This can be evaluated in a number of ways. Switching to polar coordinates, for instance,

$$= \int_0^{2\pi} \int_0^3 2r^2(\cos\theta + \sin\theta)\ dr\ d\theta = 0$$

since both $\int_0^{2\pi} \cos\theta\ d\theta = 0$ and $\int_0^{2\pi} \sin\theta\ d\theta = 0$.

C) (5) We can think of a vector field on $\mathbf{R}^2$ as a vector field on $\mathbf{R}^3$ by making the last component equal to *zero* at all points. Compute curl($\mathbf{F}$) for $\mathbf{F} = (M(x,y), N(x,y), 0)$ and relate your result to the statement of Green's Theorem.

Solution: We have

$$\text{curl}(\mathbf{F}) = \nabla \times (M(x,y), N(x,y), 0)$$

Since M, N do not depend on z and because of the zero in the last component,

$$\text{curl}(\mathbf{F}) = (0, 0, N_x(x,y) - M_y(x,y))$$

The integrand in the double integral in Green's theorem is just the last component here.

Comment: In fact, Green's theorem is a special case of a more general result called *Stokes'* theorem in which the plane region is replaced by a surface S in $\mathbf{R}^3$. Stokes' theorem then asserts the equality of the line integral of $\mathbf{F}$ over the boundary of S and the surface integral of curl($\mathbf{F}$) over S. The surface integral involves integrating curl($\mathbf{F}$) $\cdot$ $\mathbf{N}$. where $\mathbf{N}$ is a normal vector to the surface. The normal vector for D in

$\mathbf{R}^2$ is just $\mathbf{k} = (0,0,1)$ at each point, and $\text{curl}(\mathbf{F}) \cdot \mathbf{k}$ is exactly the integrand of the double integral in Green's theorem.

VIII. A function $f(x,y)$ is said to be *harmonic* on an open set U in $\mathbf{R}^2$ if it satisfies the equation

$$f_{xx} + f_{yy} = 0$$

at all points in U.

A) (5) How does a nondegenerate critical point of a harmonic function fit into our classification? Is it a local maximum, local minimum, or a saddle point? Explain how you can tell from the Hessian criterion.

Solution: Every nondegenerate critical point of a harmonic function is a *saddle point* because the Hessian matrix is

$$D^2(f) = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{pmatrix} = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{xy} & -f_{xx} \end{pmatrix}.$$

The determinant here is $-(f_{xx})^2 - (f_{xy})^2 < 0$.

B) (5) If f is harmonic, what is true about the line integral of the vector field

$$\mathbf{F}(x,y) = (-f_y, f_x)$$

around any simple closed curve in U?

Solution: Let D be the region bounded by the simple closed curve. By Green's theorem, the integral is equal to

$$\int\int_D (f_x)_x - (-f_y)_y \; dA = \int\int_D f_{xx} + f_{yy} \; dA = 0.$$

Extra Credit (20)

Suppose you follow a flow line of the vector field ∇f for $f(x,y)$ in the xy-plane. As you traverse the flow line in the increasing t-direction, is the corresponding path on the graph $z = f(x,y)$ going uphill or downhill? Explain. What does the vector field ∇f look like near a local maxmimum of f? near a local minimum of f?

Solution: You are always going *uphill* by the most direct route – recall $\nabla f(a,b)$ gives the direction in which f is increasing the fastest. The gradient vector field near a local maximum will have all arrows pointing in toward the critical point (flow lines will converge toward the maximum). Near a local minimum, the gradient vector field will be pointing away from the cririical point (flow lines will be diverging away from the minimum).

Have a peaceful and joyous holiday season!