MONT 104N – Modeling The Environment
Solutions for Final Examination – December 13, 2012

I. Wind power has emerged as the fastest growing source of energy for electrical power generation in recent years. In 2004, the generating capacity of all wind turbines in use was about $47,600$ megawatts and the generating capacity was increasing at about 26.8% per year.

A. (10) The typical English unit of power is the horsepower. 1 horsepower = .0007457 megawatts. Convert $47,600$ megawatts to the equivalent number of horsepower.

Solution: Since 1 horsepower = .0007457 megawatts, 1 megawatt = $1/.000747 \doteq 1341$ horsepower. Then

$$47600 Mw = 47600 \times 1341 \doteq 6.383 \times 10^7 \text{ horsepower.}$$

B. (10) Using the information above, construct an exponential model for WP = wind power generation as a function of t = years since 2004. *Use units of* 10^4 *megawatts for WP.*

Solution: From the given information, $r = .268$, so $M = 1 + r = 1.268$ and the exponential model is

$$WP = 4.76(1.268)^t.$$

C. (15) Fill in the table of values for WP below with values predicted by your model for the years $2004 - 2011$. Round to 2 decimal places. In what year did WP reach approximately double the 2004 level?

Solution:

$Year$	2004	2005	2006	2007	2008	2009	2010	2011
WP	4.76	6.04	7.65	9.70	12.31	15.60	19.78	25.09

WP reached approximately double its 2004 value in 2007. (Note that the exact doubling time is the solution of $1.268^t = 2$, or $t = \log(2)/\log(1.268) \doteq 2.92$ years.)

D. (10) How many years will it take for wind power generation to reach $320,000$ megawatts according to your model?

Solution: Expressing in units of 10^4 megawatts, we want to solve $32 = 4.76(1.268)^t$, or $t = \log(6.722)/\log(1.268) \doteq 8.025$. About 8 years.

E. (5) The following graph (produced by the Global Wind Energy Council – GWEC) shows the actual global wind electrical power generation capacity (estimated via surveys of electrical power producers). How do the actual figures compare with your model values? Note: The vertical scale of the graph is in *gigawatts*. 1 gigawatt = 1000 megawatts.

Solution: To compare the values, we want to multiply the values in part C above by 10 (since 10^4 megawatts = 10 gigawatts). The values are all quite close. For instance, the 2010 model prediction is 197.8 gigawatts, while the actual value was 197.6 gigawatts.

II. According to the United Nations Food and Agriculture Organization, in 2000, forest area covered 4.038×10^9 hectares of the Earth's surface. The forest area in 2010 was 4.033×10^9 hectares. Assuming that the decrease in forest area is linear, and that it will continue at the same rate into the future, in this problem you will develop a linear model for the forest area $FA =$ (in units of 10^9 hectares) remaining as a function of $t =$ years since 2000.

A. (10) Determine the slope for the linear model of the forest area.

Solution: The slope is

$$m = \frac{(4.033 - 4.038) \times 10^9}{2010 - 2000} \doteq -.0005 \times 10^9$$

hectares per year.

B. (10) What is the linear equation modeling the forest area as a function of $t =$ years since 2000.

Solution: The model is

$$FA = -.0005t + 4.038$$

C. (10) Use your equation to predict the amount of forest area that will remain in 2020.

Solution: The year 2020 is $t = 20$ years after 2000. So the prediction is

$$FA = (-.0005)(20) + 4.038 = 4.028(\times 10^9 \text{ hectares})$$

D. (5) According to your model, in what year will the forest area reach 4.0×10^9 hectares?

Solution: We solve for t in the equation

$$4.0 = (-.0005)t + 4.038 \Rightarrow t = \frac{4.0 - 4.038}{-.0005} = 76 \text{ years}$$

III. Suppose that a population of fast-reproducing insects in an area has a natural growth rate of 7% per month from births and deaths, and that there is a net migration *loss* of 100 individuals per month.

A. (5) Which of the following difference equation models for $P(n) =$ population in month n fits the description above? (Place a check next to the correct one.)

1) _____ $P(n) = 7P(n-1) - 100$

2) _____ $P(n) = 1.07P(n-100)$

3) __X__ $P(n) = 1.07P(n-1) - 100$

4) _____ $P(n) = 1.07P(n-1) + 100$

B. (10) Using an initial value $P(0) = 500$, determine the populations in months $1, 2, 3, 4, 5$ according to the model you picked in part A and record the values in the following table (round any decimal values to the nearest whole number)

n	0	1	2	3	4	5
$P(n)$	500	435	365	291	211	126

(Note: These values were computed from the difference equation in A 3) above. I kept the decimal places each time and rounded the result for the table value, but I used the decimal places to compute the next value. Your values would differ slightly if you used the rounded values. Either way is OK.)

C. (10) What happens to the population in the long run as n increases? Does it tend to a definite value? What is that value?

Solution: This is an affine difference equation and the equilibrium value is the solution of $E = 1.07E - 100$, or $E = 1428.6$. However that is an unstable equilibrium since $M = 1.07 > 1$. Mathematically, the values of the solution equation would decrease indefinitely with n. When $p(n)$ reaches 0, though, the population would have disappeared entirely (and negative values would not correspond to realistic population levels).

IV. Answer any three of the following four questions (only the best three will be counted if you answer more than three).

A. (10) What does the correlation coefficient r (or its square r^2) measure? How did we use it? Explain what it would mean, for instance if $r^2 = 1$ or $r^2 = 0$.

Solution: r or r^2 measure the degree to which one quantity y depends linearly on another quantity x in an (x, y) data set. The value $r^2 = 1$ would indicate perfect linearity (with positive slope if $r = 1$ and negative slope if $r = -1$). The value $r = 0$, would indicate no linear trend at all. We used this to measure "goodness of fit" for linear models first, and then by extension to measure "goodness of fit" for exponential and power law models, since those become linear after a transformation on the data.

B. (10) If you are fitting a power law model to a data set (x_i, y_i) "by hand," you would start by transforming the data to $(X, Y) = (\log(x_i), \log(y_i))$. If the best fit regression line for the transformed data is $Y = mX + b$, what is the corresponding power law model? (Assume the logarithms have base 10 as we discussed in class.)

Solution: If $\log(y) = m\log(x) + b$, then exponentiating both sides with base 10, we get $y = 10^b x^m$. This is the power law model.

C. (10) A population has unrestricted growth rate $r_{max} = .03$ and carrying capacity $K = 1000$. What is the corresponding logistic model?

Solution: The model is given by

$$p(n) = 1.03p(n-1) - \frac{.03}{1000}(p(n-1))^2$$

D. (10) The following graph shows W = the word production of photovoltaic arrays (used for solar power generation) in units of "peak megawatts." Between 1998 and 2007, what type of model would be most appropriate for describing how W is growing. Explain. *Look at the vertical axis scale carefully!*

Solution: The vertical axis scale is logarithmic (because multiplying y by a factor of 10 moves a fixed distance along the axis). Since that portion of the graph is very close to linear, the relation is $\log(W) = mt + b$, so $W = (10^b)(10^m)^t$. This is an *exponential model.*

V. Essay. (60) In general terms, what is a mathematical model? Describe what they are, how they are constructed, and how they are used. Give examples of two different types of mathematical models we have studied in this course. Next, *why* do we try to build mathematical models of aspects of the real world? Can *any* mathematical model be a completely accurate representation of some aspect of the natural world? As an example, why do scientists think it is important to understand how much CO_2 is present in the atmosphere? What tends to happen when CO_2 levels rise? Describe a key piece of evidence that suggests human activities might have changed atmospheric CO_2 levels over the past 50-200 years. Explain the case for saying the evidence points to that conclusion, and relate your answer to the results of modeling exercises we did in this class.

Model Answer: A mathematical model is an equation, a system of equations, a diagram, or a relationship expressed in mathematical terms that aims to capture some aspect of the behavior of a real-world system. They are usually constructed by analyzing data from observations or experiments and "fitting" a mathematical form to the observed relationship between the measured quantities. We have studied *function models* including linear, exponential, and power law forms, and difference equation models in this course. As mentioned above, models are usually used to develop some understanding of the real-world system in cases or situations where data has not been collected. They are often used to interpolate or extrapolate values for quantities in cases where data has not been collected, for instance to make predictions about future behavior of the system. No mathematical model can reflect every aspect of a real world system, though, since the model-building process always leaves out something.

Modeling CO_2 levels in the atmosphere is of importance because CO_2 is a greenhouse gas. It tends to keep solar radiation reflected from the Earth's surface from passing back out into space, so the higher the CO_2 level is, the higher the average surface temperature will get (other things being equal). This has secondary effects such as rising sea levels, more severe weather patterns, and long-term climate change.

One key piece of evidence for why human activities may have changed CO_2 levels over the past 50-200 years is the measurements made at the Mauna Loa observatory in Hawaii since the 1950's. In one of our projects we analyzed the monthly average measurements in this data set and we saw that, after factoring out a predictable annual seasonal variation (due to plant activity taking up CO_2 through the summer months in the northern hemisphere), the CO_2 levels measured at Mauna Loa have been steadily increasing by about 2.0 ppm per year in recent years. (We also saw in our project that an exponential model with an annual growth rate around .5% gave a marginally better fit than a linear model, but in real terms the difference was not very large over the short run.) Since humans only began putting large quantities of CO_2 into the atmosphere around the time of the Industrial Revolution in the mid-1800's, and there have not been frequent volcanic eruptions or other occurrences that would explain the CO_2 concentration increases over this period, these measurements are often taken as evidence that humans have had an effect on the atmosphere.