This Paper is Incomplete

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In the 20th century, the mathematician and philosopher Kurt Gödel made one of the greatest mathematical discoveries of all time. Through his Incompleteness Theorem, Gödel proved that it is impossible to prove certain propositions within a given system while showing that arithmetic is both inconsistent and incomplete. His work is seen in many analogies and metaphors and can be compared to the work of the artist M.C. Escher as well as the musician J. S. Bach. The theorem has also shown the superiority of the human mind over that of computers while changing the way mathematics can be viewed.

Kurt Gödel was born on April 28th, 1906 in Brünn, Austria-Hungary (now Brno, Czech Republic). His father, Rudolf Gödel, was a managing director and part owner of a major textile firm in Brünn. His mother, Marianne Handschuh, was 14 years Rudolf’s junior and possessed some literary education, having undertaken part of her studies in France.[[1]](#footnote-1) For the most part, Gödel seems to have had a happy childhood, living with his father, mother, and brother, Rudolf, in their family home in Brünn. He was rather devoted to his mother, becoming very timid and troubled whenever his mother was not present in their home. Around the age of six, he came down with rheumatic fever, which seemed to come and go without any major effect on Gödel. It was not until age eight that he began reading medical books about his illness and its symptoms. Amongst the many symptoms and after effects from having the disease, he found that a weak heart was a possible complication that came from rheumatic fever. While there is no proof as to whether Gödel had a weak heart or not, his research seemed to have convinced him of the former. His health and well-being became an everyday worry, an obsession that would plague him until the end of his life.

During his youth, Kurt Gödel attended school in Brünn, completing his school studies in 1923. Schoolwork seemed to come easily to Gödel, with his brother Rudolf remarking that,

“Even in High School my brother was somewhat more one-sided than me and to the astonishment of his teachers and fellow pupils had mastered university mathematics by his final [Gymnasium](javascript:win1('../Glossary/gymnasium',350,200)) years. ... Mathematics and languages ranked well above literature and history. At the time it was rumored that in the whole of his time at High School not only was his work in Latin always given the top marks but that he had made not a single grammatical error.”[[2]](#footnote-2)

Gödel entered into the University of Vienna later that year, torn between specializing in mathematics and theoretical physics. His professors made an impact on the young Gödel, but none more so than Philipp Furtwängler, his mathematics professor. While Furtwängler was a great mathematician, teacher, and lecturer, perhaps it was his own medical issues that drew a connection from Gödel. Paralyzed from the neck down, Furtwängler required an assistant to write on the blackboard while he gave his lectures from a wheelchair, something that might have swayed the health-conscious Gödel into becoming a mathematician. He finished his doctoral dissertation in 1929 and became a member of the University of Vienna’s faculty, teaching in the school of logical positivism until 1938.[[3]](#footnote-3) It was during his time teaching at the University that Gödel devised one of the most important proofs of the 20th century: the Theorem of Incompleteness.

The Theorem of Incompleteness is perhaps Kurt Gödel’s most important proof. He published the results of his research in 1931, under the title of *Über formal unentscheidbare Sätze der Principia Mathematica und verwandter Systeme* (On Formally Undecidable Propositions of Principia Mathematica and Related Systems).[[4]](#footnote-4) In his paper, Gödel proved the fundamental results of axiomatic systems and showed that, in any axiomatic mathematical system, there are propositions that cannot be proved or disproven within the axioms of the system. Such a discovery ended a hundred years of attempts to establish axioms, putting the whole of mathematics on an unquestionable base. Gödel’s results were also a major landmark in 20th century mathematics, mainly due to proving that, contrary to popular belief, mathematics is not a finished object. Gödel submitted the theorem for his habilitation in 1932 and received the title of Privatdozent at the University of Vienna almost a year later.

As a brilliant mathematician with a promising future, Gödel did not limit himself to one country. In 1934, Gödel traveled to the U.S., where he gave a series of lecture at Princeton entitled “*On undecidable propositions of formal mathematical systems*.”[[5]](#footnote-5) However, on his way back to Europe, he suffered from a nervous breakdown and telephoned his brother Rudolf, who had become a doctor, to say that he was staying in Paris to recover. Yet despite needing treatment from a psychiatrist to recover, his research was going along smoothly. In 1935, he had made important discoveries regarding the consistency of the axiom of choice along with the other principles of set theory. However, this was not to last; a National Socialist student murdered Moritz Schlick, whose seminar had aroused Gödel’s interest in logic. Gödel was very much affected by Schlick’s death and had another breakdown in 1936. After this, he returned to Vienna to marry Adele Porkert in the autumn of 1938, who was six years older than Gödel, had been married once before and both his parents, particularly his father, objected to the idea of their marriage. But the disagreement about whom he was marrying would seem small compared to the events that were about to unfold.

In 1933, about the time Gödel received the title of Privatdozent at the University of Vienna, Adolf Hitler came to power. At first, this seemed to have no effect of Gödel; he was little interested in politics and was far too engrossed in his studies to care. While Gödel was returning to Vienna after his latest series of nervous breakdown, Austria had officially become a part of Germany. After Austria joined Germany, most people who held the title of Privatdozent became paid lecturers. Gödel’s application, on the other hand, was given a rather unenthusiastic response. This very likely stemmed from the fact that most people thought that he was Jewish, due to a large number of his friends being Jewish. While this was not actually the case, the belief was bad enough to the point that he was once attacked by a gang of youths while out walking with his wife through the streets of Vienna. When the war started, Gödel feared that he might be conscripted into the German army. Although he was convinced that he was in far too poor health to serve in the army, he feared that if he could be mistaken for a Jew, he could just as easily be mistaken for a healthy man as well. Unwilling to take this risk and after lengthy negotiation to obtain a U.S. visa, he was fortunate to be able to return to the United States with his wife, although he would have to travel through Russia and Japan to do so. In 1940, Gödel and his wife arrived in the United States, where he would continue his research without the war hanging over him.

Kurt Gödel transitioned easily into his new life in the United States. He was an ordinary member of the Institute for Advanced Study from 1940 to 1946, where he then became a permanent member until 1953. He also returned to his roots as a professor, holding a chair at Princeton from 1953 until his death, although he had a contract with the school that explicitly stated that he had no lecture duties.[[6]](#footnote-6) While at Princeton, Gödel was close friends with Albert Einstein, a German-born theoretical physicist who is regarded as the father of modern physics. As a mathematician-turned-physicist (Einstein) and a physicist-turned-mathematician (Gödel), each man held the other one in the highest esteems and they spoke frequently, with Gödel even finding new solutions to Einstein’s field equations.[[7]](#footnote-7)

While at Princeton, Gödel received numerous awards and honors for his work, receiving the Einstein Award in 1951 and the National Medal of Science in 1974. It was also during this time that he was inducted into numerous international institutes, such as the [National Academy of Sciences](http://www-history.mcs.st-andrews.ac.uk/Societies/NAS.html) of the United States, the [Royal Society](http://www-history.mcs.st-andrews.ac.uk/Societies/RS.html), the Institute of France, the Royal Academy, and the [London Mathematical Society](http://www-history.mcs.st-andrews.ac.uk/Societies/LMS.html).[[8]](#footnote-8) Ironically, the one country that he refused to be associated with was his native Austria. He turned down membership into the Academy of Sciences and again when he was awarded honorary membership. He also refused to accept the highest National Medal for scientific and artistic achievement that Austria offered him. He certainly felt bitter at his own treatment during the World War II era, but equally so about the treatment that his own family had received. In 1944, in terms of the treaty negotiated after the war between the Austrians and the Czechs, Gödel’s mother received one tenth of the value for her villa in Brno; an injustice that infuriated Gödel to no end. In fact he always took such injustices as a personal affront, despite the fact that a large number of families suffered in the same way.

After he had been made an U.S. citizen in 1948, Gödel continued to produce work of great importance to the mathematical world. His masterpiece, *Consistency of the axiom of choice and of the generalized* [*continuum-hypothesis*](javascript:win1('../Glossary/continuum_hypothesis',350,200)) *with the axioms of set theory*, (1940) is a classic of modern mathematics. In the theory, Gödel proved that, if an axiomatic system of set theory of the type proposed by Russell and Whitehead in *Principia Mathematica* is consistent, then it will remain so when the axiom of choice and the generalized continuum-hypothesis are added to the system.[[9]](#footnote-9) This means that, 1) Any system as powerful as number theory, which can prove its own consistency, that system is necessarily inconsistent and 2) Any system as powerful as number theory, which can prove its own consistency, that system is necessarily incomplete. Unfortunately, this would be one of the last theorems he would produce before his death.

Towards the end of his life, Gödel became increasingly worried about his health. His brother, Dr. Rudolf Gödel, wrote that:

My brother had a very individual and fixed opinion about everything and could hardly be convinced otherwise. Unfortunately he believed all his life that he was always right not only in mathematics but also in medicine, so he was a very difficult patient for doctors. After severe bleeding from a duodenal ulcer ... for the rest of his life he kept to an extremely strict (over strict?) diet which caused him slowly to lose weight.[[10]](#footnote-10)

To make matters worse, Gödel became convinced that attempts were being made to put poison in his food. In a rather ironic twist of fate, he refused to eat in order to avoid being poisoned, essentially starving himself to death. His wife, Adele, did what she could to support his decisions and to ease the tensions that constantly troubled him. However, she began to suffer her own health problems, having endured two strokes and a major operation. Gödel died on January 14th, 1978, “sitting in a chair in his hospital room at Princeton”.[[11]](#footnote-11) And with that, one of the brightest lights in 20th century mathematics faded to black.

Douglas Hofstadter is the author of *Gödel, Escher, Bach: An Eternal Golden Braid*, a book where he illustrates ideas and themes that all three artists use in their works. The book itself has many uses of the themes within the writing. For example, Hofstadter uses recursion and acrostic in dialogues in the book. For reference, Maurits Cornelis Escher was a Dutch artist and lived from June 17, 1898 to March 27, 1972. Known as MC Escher, he was an artist and was well known for his use of mathematics within the art. Infinity, tessellation, and paradoxes were often subjects of Escher’s works.[[12]](#footnote-12) Johann Sebastian Bach was a composer and organist and lived from March 21, 1685 to July 28, 1750.[[13]](#footnote-13) He too incorporated mathematics in his musical pieces.

Gödel, Escher, and Bach all use recursion in their works. Recursion is defined as an expression such that each term is generated by repeating a particular mathematical operation[[14]](#footnote-14). Classic examples of the use of recursion in mathematics are the Sierpinski triangle where there are an infinite number of triangles in a triangle, factorials which mean n! = n(n-1)!, or the Fibonacci sequence.

Music has a recursive structure, for themes and harmonies are embedded inside a piece of music between keys and scales. It is possible to play a theme within one key and play another theme inside another key creating a recursive structure. Bach utilizes recursion in his piece *Crab Canon*. The piece of music can be played upside down and backwards and will still be the same was when it is played forward.As a result of Bach’s influence, Escher created a piece also called *Crab Canon* where he tessellated two different colored crabs without gaps between the crabs. Escher “made one single theme mesh with itself going both backwards and forwards.”[[15]](#footnote-15) Bach also uses recursion in a piece of his entitled *Little Harmonic Labyrinth* where “he tries to lose you in a labyrinth of quick key changes.”[[16]](#footnote-16) The application of Gödel’s incompleteness theorem shows recursion, for when a part of a problem requires another problem outside of the system, recursion is used. Each time you leave the system, it’s an application of recursion. The statement that results from Gödel’s theorem – this statement is not provable – is an example of self-reference because it uses the word “this” to refer to the sentence. Because it references itself and “pushes” out of the statement to another, it exemplifies recursion. Another example of recursion is in the game of chess, for there is a recursive tree of chess plays. Players think of moves to do with each individual piece and what the move could result in, etc., but ultimately the thoughts are reverted back to what the move will be. Each new result is a “push” from the first thought or a “pop” back, words often used in computer programming and in Hofstadter’s book.[[17]](#footnote-17)

Both paradox and strange loop plays into the works of Gödel, Escher, and Bach. Paradox is a statement that contradicts itself.[[18]](#footnote-18) Strange loop is when

…there is a shift from one level of abstraction (or structure) to another, which feels like an upwards movement in a hierarchy, and yet somehow the successive "upward" shifts turn out to give rise to a closed cycle. That is, despite one's sense of departing ever further from one's origin, one winds up, to one's shock, exactly where one had started out. In short, a strange loop is a paradoxical level-crossing feedback loop.[[19]](#footnote-19)

A classic example of paradox is the Zeno’s Paradox. Zeno’s Paradox is the idea that it is impossible to get from one place to another, for when you go from one place to another you must go half the distance, then half of that, then half of that, and so on. It seems to show that motion is impossible or an illusion, although we know it is possible to get from one place to another which makes it a paradox.

In Escher’s piece *Drawing Hands*, the two hands draw each other’s hand is a perfect example of the idea of the strange loop. The right hand is drawing the left hand, and the left hand is drawing the right hand creating a circle and a paradoxical thought of who is actually drawing the hands. There is a two-step strange loop, because there is a tangled, visual hierarchy of the hands drawing each other, and there is an inviolate, invisible level where Escher is drawing the hands. [[20]](#footnote-20) Escher’s *Waterfall* is another example of strange loop. The water starts at one point and ends at the starting point by going upwards on a path in six steps making the piece a six-step strange loop. Other pieces of Escher’s like *Ascending and Descending* are many more steps and are “loose” strange loops.[[21]](#footnote-21) Each of these pieces exhibits the idea of infinity, because there is no concrete beginning or end. Bach’s piece entitled “Canon per Tonos,” his never-ending canon, holds a paradox by going up a step each of the six repeats and returning to the origin plus an octave without the listener realizing the changes occurred. The process repeats ad infinitum. It does seem that since the piece goes up a step, it should never go back to where it started, but it does. “Through the levels of some hierarchical system, we unexpectedly find ourselves right back where we started” creating a strange loop.[[22]](#footnote-22)

Consistency is an integral part of Gödel’s Theorem. In Escher’s piece *Relativity*, the stairs look possible to walk on, but the way the image is orientated does not seem possible. Sometimes, parts of the whole can be possible, but the whole cannot be possible, which creates inconsistency within the system.

In addition to writing *Gödel, Escher, Bach: An Eternal Golden Braid*, Douglas Hofstadter also wrote an article that serves as a means of better explaining Gödel’s Theorem of Incompleteness using several metaphors and paradoxical sentences. One example of Hofstadter’s paradoxical wordplay is in the sentence: I never make misteaks.[[23]](#footnote-23) The reason this is a paradox is because the writer of the sentence claims to never make mistakes. However, because the writer misspells mistakes as “misteaks,” this proved the sentence to be false. In addition, Hofstadter presents his own paradoxical statement in the form of Hofstadter’s Law. In it, he states that: “It always takes longer than you expect, even when you take into account Hofstadter’s Law.”[[24]](#footnote-24) In other words, even when you take Hofstadter’s Law into account, it will always take you longer to do something that you expected it to. Hofstadter also proves an interesting paradox about history, stating that, “One of the lessons of history is that no one ever learns the lessons of history.”[[25]](#footnote-25) This statement only further proves the tragic fact that those who do not learn history are doomed to repeat it.

Perhaps the best example Hofstadter gives in his article is the Epimenides Paradox, which states that: “This sentence is false.”[[26]](#footnote-26) The statement is trapped in an endless cycle, wherein if the sentence is true, then it is false and if it is false, then it is true. Gödel’s Sentence, this sentence is not provable (in formal system X), is a more formalized means of stating the Epimenides Paradox. While many of these paradoxical statements are certainly self-contradictory, these are not the only examples Hofstadter gives to simplify Gödel’s Theorem of Incompleteness.

Along with his sentence examples, Hofstadter also uses several scenarios that also show the self-contradiction of the Theorem of Incompleteness. In his first scenario, Hofstadter presents us with the image of a laser pointer standing in a room with a dot on its back end. He goes on to point out how the only place in the room that the laser cannot hit is the black dot on its own rear end. Hofstadter provides a similar example that talks about how the only place a fly can land in a room that is safe is on the handle of the fly swatter. Both scenarios show how, you can use the laser pointer/ flyswatter to hit almost every surface in the room, but in order to hit the black dot/ fly, you need another laser pointer/ flyswatter to hot it. Similarly, according to Gödel’s Theorem, you can use an outside statement to prove that a statement is true, but then you need another statement to prove that both the statement and the outside statement are true.

Gödel’s Theorem is also explained in a “Wild Dance Version.”[[27]](#footnote-27) This version of the Theorem gives us the statement of “this record cannot be played at this party,” serving once again as a means of showing the infinite cycling of the Theorem of Incompleteness. The grooves of the record are hit by the needle, which uses electricity to play the sound through the speakers, where the sound then travels through the air, through our eardrums, and into our brains. From there, our brains send signals through the nerves in our body, causing us to dance, sing, and shout, which rock the table where the phonograph that is playing the record is sitting. To put it simply, by playing the record at the party, the additional noise is creates are so great and so numerous, that the original sound of the record cannot be heard. Likewise, if one tries to solve an equation using statements outside of the original, the outside statements and proofs become so numerous that you forget what the original equation was trying to prove.

The last example Hofstadter gives in his article is the DNA Fish, an image that shows the word “DNA” morphing into a fish with the word “DNA” inside of it. This is meant to represent the way in which self-reference can be achieved through mathematical equations. The DNA codes not only for the fish, but for itself as well, similar to how the Theorem of Incompleteness forces equations to be proven in statements outside of itself that, in turn, include it. Through these examples, Hofstadter shows us how, with the Theorem of Incompleteness, no equation can ever fully be proven.

Kurt Gödel’s greatest contribution to mathematics was his Incompleteness Theorem. It first appeared in 1931 in a German scientific periodical called “On Formally Undecidable Propositions of Principia Mathematica and Related Systems.” At only 25 years old, Gödel produced a paper that became a landmark in the history of not only mathematics, but logic as well.[[28]](#footnote-28) This paper found a central problem in the foundations of mathematics. From the time when Euclid wrote his famous work, *The Elements*, mathematicians have tried to deduce everything in mathematics down to a basic set of axioms. This is the idea of accepting certain truths without proof. Gödel, however, showed that this is impossible. He showed that there are limitations that prevent the possibility of establishing logical consistency within a system, such as arithmetic, without using principles that are so complex that their consistency is as doubtful as the systems themselves. This makes it impossible to prove that any branches of mathematics are entirely free from internal contradictions.[[29]](#footnote-29) There are several developments in mathematics and logic that helped Gödel reach this conclusion.

The first question that arises in deducing a proof is whether a set of postulates is internally consistent such that no contradictory theorems can be deduced from the same set of postulates.[[30]](#footnote-30) This makes one wonder whether or not the Euclidean axioms are true and consistent. As it is, they cannot be proven but are nonetheless accepted. An absolute proof is the only way to establish perfect consistency. Nagel and Newman define it as something that, “achieves its objectives by using a minimum of principles of inference, and does not assume the consistency of some other set of axioms.”[[31]](#footnote-31) The first step in such a proof is the complete formalization of a deductive system. This is the process of removing all meaning from the expressions occurring within the system.[[32]](#footnote-32) Formalization ultimately presents the basic functions and structure behind the proof. Within these formalized systems lie the meta-mathematical statements that are prevalent within Gödel’s Proof. These statements show the relationship between formulas and axioms.

Gödel determined that given any consistent set of arithmetical axioms, there are true arithmetical statements that can be derived from the set.[[33]](#footnote-33) He was able to reach this conclusion through usage of the Richard Paradox, an idea concluded by a French mathematician named Jules Richard in 1905. To start, one must consider a meta-mathematical language like English in which it is possible to make definitions and statements about integers. From here each definition must be listed in a finite order by number of letters going in an increasing order. Each definition is then numbered in the order it appears on the list. This is where recursion comes into the idea. If the number of the definition is not the number that is being defined, then that number is Richardian. This results in a paradox because the number 15 does not equal the list’s number 15, for example.[[34]](#footnote-34)

The idea of mapping is the final concept that needs to be understood to fully understand Gödel’s Proof. Mapping allows geometry to be translated into algebra and also plays an important role in physics, such as the representation of currents. In Gödel’s case, mapping is used to relate objects in different domains. The concept of mapping relates to the idea of mirroring.[[35]](#footnote-35)

Once understanding the problem with consistency, the idea of mapping, and the Richard Paradox, Kurt Gödel’s proof can be comprehended. Gödel begins by assigning a unique number to each elementary sign, each formula, and each proof. These numbers came to be known as “Gödel numbers.”[[36]](#footnote-36) For example:

( Ǝ *x* ) ( *x* = *s y* ) 8 4 11 9 8 11 5 7 13 9

Each of these numbers is the power to which each successive prime number is raised. The formula should be referred to as *m*. The same process is then followed for a second formula. In this case the formula should be referred to as *n*. By using the ideas of mirroring, it is now possible to reach the formula *k*= (2^*m*) (3^*n*). This number equals 243,000,000, though in terms of Gödel numbers it states that 0 equals 0.[[37]](#footnote-37) At this point, the meta-mathematics becomes completely arithmetized.

Gödel’s argument shows that it is possible to construct the arithmetical formula G that represents a meta-mathematical statement saying that G is not demonstrable. This means that G says of itself that it is not demonstrable. Because both a formula and its negation are demonstrable, the arithmetic is not consistent. However, it is a true arithmetic formula. Therefore the axioms of arithmetic must be incomplete. This means that not all arithmetic truths can be deduced from the axioms.[[38]](#footnote-38)

The Incompleteness Theorem leads to several conclusions. First, it demonstrates that there are an endless number of true arithmetical statements which cannot be formally deduced from any given set of axioms by a closed set of rules of inference.[[39]](#footnote-39) Ernest Nagel and James R. Newman also bring to mind the question of whether any sort of calculating device could ever match the human brain in terms of mathematical intelligence. Calculating machines contain a fixed set of directives based on fixed rules of inference. However, because Gödel showed that there are an infinite number of problems that fall outside of a fixed method, machines cannot solve them, but the human brain is able to comprehend them and use intuition. They feel that the Incompleteness Theorem shows that, “the structure and power of the human mind are far more complex and subtle than any non-living machine yet envisaged.”[[40]](#footnote-40)

In addition to his Incompleteness Theorems, Gödel made several other significant discoveries. He was able to prove that there are not a finite number of axioms.[[41]](#footnote-41) He also made significant contributions to proof theory, model theory, recursion theory, set theory, and intuitionist logic.[[42]](#footnote-42) Interestingly, not all of his discoveries and proofs were related to mathematics. When Gödel went to become an American citizen in 1947, he took his studying and preparation very seriously. He studied the United States’ Constitution so thoroughly that he discovered an internal contradiction that would allow American democracy to deteriorate into tyranny. On the day of his exam, his close friend Albert Einstein warned him not to bring up this contradiction because it could ultimately jeopardize his citizenship.[[43]](#footnote-43) Gödel was also fascinated with God and spent a large portion of his life trying to derive a proof of His existence. Many great philosophers had tried to tackle this ontological proof in the past including Saint Anselm, René Descartes, and Gottfried Leibniz. As it turned out, he died before he was able to complete the proof with only one step to its completion.[[44]](#footnote-44) Like so many other discoveries of his, this proof was “incomplete.”

In sum, the Incompleteness Theorem has had a huge impact on mathematics today. However, many mathematicians disregard it and continue to perform math as the way it has always been in Euclidian fashion. Like the works of Escher and Bach, Gödel’s proof stimulates the mind and changes people’s perceptions. It was a major breakthrough in mathematics that may never get the recognition it deserves. In the 20th century,

the mathematician and philosopher Kurt Gödel made one of the greatest mathematical discoveries of all time.Bibliography

“Biography.”  *M.C. Escher: The Official Website*. The MC Escher Company.  2011. Web. 07 May 2011. < http://www.mcescher.com/>.

Goldstein, Rebecca. *Incompleteness: The Proof and Paradox of Kurt Gödel.* New York: W.W. Norton and Company, 2005. Print.

Hanford, Jan. “Biography.” *J.S.Bach*. 2011. Web. 07 May 2011. <http://www.let.rug.nl/Linguistics/diversen/bach/map.html>.

Hofstadter, Douglas R. "Analogies and Metaphors to Explain Gödel’s Theorem." Mathematical Association of America. 13.2 (1982): 98-114. Print.

Hofstadter, Douglas R. *Gödel, Escher, Bach: An Eternal Golden Braid*. New York: Basic Books. 1979. Print.

Hofstadter, Douglas R. *I Am a Strange Loop*. New York (N.Y.): Basic, 2007. Print.

Nagel, Ernest, and James R Newman. *Gödel’s Proof*. New York: New York University Press, 1958. Print.

O'Connor, J.J., and E.F. Robertson. "Kurt Gödel."Histories. School of Mathematics and Statistics University of St Andrews, Scotland, Oct 2003. Web. 6 Apr 2011. <http://www-history.mcs.st-andrews.ac.uk/Biographies/Godel.html>.

“Paradox.” *WordNet.* Princeton.edu. 2011. Web. 15 Apr 2011. <wordnetweb.princeton.edu/perl/webwn >.

“Recursion.” *WordNet.* Princeton.edu. 2011. Web. 15 Apr 2011. <wordnetweb.princeton.edu/perl/webwn >.

Wang, Hao. *Reflections on Kurt Gödel*. Cambridge: The MIT Press, 1987. Print.

1. O'Connor, J.J., and E.F. Robertson. "Kurt Gödel."Histories. School of Mathematics and Statistics University of St Andrews, Scotland, Oct 2003. Web. 6 Apr 2011. <http://www-history.mcs.st-andrews.ac.uk/Biographies/Godel.html>. [↑](#footnote-ref-1)
2. O'Connor, J.J., and E.F. Robertson. "Kurt Gödel."Histories. School of Mathematics and Statistics University of St Andrews, Scotland, Oct 2003. Web. 6 Apr 2011. <http://www-history.mcs.st-andrews.ac.uk/Biographies/Godel.html>. [↑](#footnote-ref-2)
3. Ibid [↑](#footnote-ref-3)
4. Ibid [↑](#footnote-ref-4)
5. O'Connor, J.J., and E.F. Robertson. "Kurt Gödel."Histories. School of Mathematics and Statistics University of St Andrews, Scotland, Oct 2003. Web. 6 Apr 2011. <http://www-history.mcs.st-andrews.ac.uk/Biographies/Godel.html>. [↑](#footnote-ref-5)
6. O'Connor, J.J., and E.F. Robertson. <http://www-history.mcs.st-andrews.ac.uk/Biographies/Godel.html>. [↑](#footnote-ref-6)
7. Wang, p. xxiii, p. 30-31 [↑](#footnote-ref-7)
8. O'Connor, J.J., and E.F. Robertson. <http://www-history.mcs.st-andrews.ac.uk/Biographies/Godel.html>. [↑](#footnote-ref-8)
9. O'Connor, J.J., and E.F. Robertson. <http://www-history.mcs.st-andrews.ac.uk/Biographies/Godel.html>. [↑](#footnote-ref-9)
10. Ibid [↑](#footnote-ref-10)
11. Ibid [↑](#footnote-ref-11)
12. “Biography.”  *M.C. Escher: The Official Website*. The MC Escher Company.  2011. Web. 07 May 2011. < http://www.mcescher.com/>. [↑](#footnote-ref-12)
13. Hanford, Jan. “Biography.” *J.S.Bach*. 2011. Web. 07 May 2011. < http://www.let.rug.nl/Linguistics/diversen/bach/map.html>. [↑](#footnote-ref-13)
14. Recursion definition, (wordnetweb.princeton.edu/perl/webwn) [↑](#footnote-ref-14)
15. Hofstadter, Douglas R. *Gödel, Escher, Bach: An Eternal Golden Braid*. New York: Basic Books. 1979. Print. 207 [↑](#footnote-ref-15)
16. Gödel, Escher, Bach 138 [↑](#footnote-ref-16)
17. Gödel, Escher, Back 137 [↑](#footnote-ref-17)
18. Paradox definition, (wordnetweb.princeton.edu/perl/webwn) [↑](#footnote-ref-18)
19. Hofstadter, Douglas R. *I Am a Strange Loop*. New York (N.Y.): Basic, 2007. Print. (101-102) [↑](#footnote-ref-19)
20. Gödel, Escher, Bach 685 [↑](#footnote-ref-20)
21. Gödel, Escher, Bach 18,21 [↑](#footnote-ref-21)
22. Gödel, Escher, Bach 18 [↑](#footnote-ref-22)
23. Hofstadter, p. 101 [↑](#footnote-ref-23)
24. Ibid [↑](#footnote-ref-24)
25. Ibid [↑](#footnote-ref-25)
26. Ibid [↑](#footnote-ref-26)
27. Hofstadter, p. 107 [↑](#footnote-ref-27)
28. Ernest Nagel and James R. Newman, Gödel’s Proof (New York: New York University Press, 1958), 3 [↑](#footnote-ref-28)
29. Nagel and Newman, 6 [↑](#footnote-ref-29)
30. Nagel and Newman, 14 [↑](#footnote-ref-30)
31. Nagel and Newman, 33 [↑](#footnote-ref-31)
32. Nagel and Newman, 26 [↑](#footnote-ref-32)
33. Nagel and Newman, 58-59 [↑](#footnote-ref-33)
34. John Little, “More About Richard’s Paradox” Message to Matthew Kasuba, Astrid Ludwig, and Peter Zona. May 3, 2011. E-mail [↑](#footnote-ref-34)
35. Nagel and Newman, 63-66 [↑](#footnote-ref-35)
36. Nagel and Newman, 68-69 [↑](#footnote-ref-36)
37. Nagel and Newman, 72-76 [↑](#footnote-ref-37)
38. Nagel and Newman, 85-86 [↑](#footnote-ref-38)
39. Nagel and Newman, 98 [↑](#footnote-ref-39)
40. Nagel and Newman, 101-102 [↑](#footnote-ref-40)
41. Nagel and Newman, 56 [↑](#footnote-ref-41)
42. Rebecca Goldstein, Incompleteness : The Proof and Paradox of Kurt Gödel (New York, NY: W.W. Norton and Company, 2005), 252 [↑](#footnote-ref-42)
43. Goldstein, 232-233 [↑](#footnote-ref-43)
44. Goldstein, 209-210 [↑](#footnote-ref-44)