College of the Holy Cross
MONT 104Q – Mathematical Journeys
Solutions for Final Exam – December 18, 2015

I. G. H. Hardy included two proofs from Euclid's *Elements* as prime examples of beautiful, serious mathematics in his book *A Mathematician's Apology*. The first was the proof that there are infinitely many prime numbers.

A) (5) Prove this result using Euclid's method.

Assume that the list of all primes is finite, and that this list contains just the primes $2, 3, \ldots, P$. Then form the integer

$$Q = 2 \cdot 3 \cdot \cdots \cdot P + 1.$$

Since Q is an integer, it must be divisible by some prime number. But each of the primes in the list $2, 3, \ldots, P$ leaves a remainder of 1 when it divides Q. Therefore none of the primes in the list can divide Q and this is a contradiction. Hence the claim that there are infinitely many primes is proved.

B) (5) What is the name of the method of proof that Euclid (and you) used here? Describe briefly how that method works. Hardy say this is "a far finer gambit than any chess gambit: a chess player may offer the sacrifice of a pawn or even a piece, but a mathematician offers *the game*?" What does he mean by that?

The name of this proof technique is *proof by contradiction*, or *reductio ad absurdum*. It works like this: To prove a statement, you begin by assuming the negation of that statement (that is by assuming that the statement you want to show is *false*). Then you reason from the negation and try to produce a contradiction. This shows that the negation must be false and the statement we were trying to prove must be true. Hardy is referring to the fact that when a mathematician structures a proof this way, it can seem as though he or she is offering the whole "game" (the whole statement to be proved) by assuming that is false to begin with. He compares this to a gambit in chess, where one might offer a pawn or a piece for the opponent to capture while gaining an advantage in the process. The chess analog of the mathematical maneuver, Hardy says, would be to offer *all one's pieces* or to offer to *resign the whole game*, and then seize the advantage.

II.

(A) (10) Give the statement and proof of Proposition 29 in Book I of Euclid's *Elements*.

The statement is that if a transversal GH falls on two parallel lines AB and CD, then the alternate interior angles are equal, the corresponding angles are equal, and the interior angles on one side of the transversal add to two right angles.

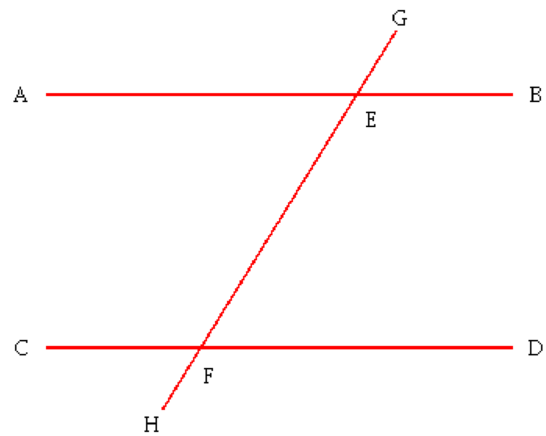

Figure 1: Figure for Proposition 29, Book I

Here's the way the proof works: Using a proof by contradiction, Assume that the first part does not hold, say that $\angle BEF \neq \angle CFE$. If this is true, one must be strictly smaller, say $\angle BEF < \angle CFE$. But then if we add $\angle DFE$ to both sides of that inequality we get (by what we called "Common Notion 6" when we discussed it – that's my name, not Euclid's by the way)

$$\angle BEF + \angle DFE < \angle CFE + \angle DFE = 2 \text{ right angles}$$

(using Proposition 13). By Postulate 5, this implies that if we extend lines AB and CD, they must meet on the side of the transversal containing B, D. But that contradicts the assumption that AB and CD are parallel lines. The remaining parts now follow from other facts proved previously. The corresponding angles $\angle BEF$ and $\angle DFH$ are equal because $\angle BEF = \angle CEB$ from the first part, but then also $\angle BEF = \angle DFG$ since vertical angles at the intersection of two lines are equal (Prop. 15). The desired statement then follows from Common Notion 1. This also implies the final part of the proposition.

(B) (5) How does this proposition relate to Proposition 27? What is special about the place of Proposition 29 in Book I? Explain briefly.

The parts of this statement are the *converses* of parts of Proposition 27. For instance, part of Prop. 27 says that if opposite interior angles at the transversal are equal, then the two lines are parallel. Proposition 29 is special in the logical development of Book I of the *Elements* because it is the first time Euclid uses Postulate 5. Everything up to this point is true even in geometries where Postulates 1 - 4 are true, but Postulate 5 does not hold.

III. (10) What does it mean to say that a polyhedron is *convex*? What is true about the number of vertices V, the number of edges E, and the number of faces F in that case?

A polynohedron X is convex if for all points P, Q contained in X, the straight line segment from P to Q is completely contained in X.

The relation between V, E, F is, of course the one discussed in *Proofs and Refutations*, the Euler relation:

$$V - E + F = 2.$$

IV. In our proof of the "power theorem" about the remainders $r^{m-1}\ R\ m$, where m is prime and $1 \le r \le m-1$,

(A) (5) The first step was to show that for each fixed r as above, the remainders $r\ R\ m$, $(2 \cdot r)\ R\ m$, ... , $((m-1) \cdot r)\ R\ m$ were all distinct. Show this is true. You may assume the statement of Euclid's lemma about primes: if a prime m divides a product of integers $c \cdot d$, then m divides c or m divides d.

Arguing again by contradiction, suppose that $(a \cdot r)\ R\ m = (b \cdot r)\ R\ m$ for some $1 \le a < b \le m-1$. By general properties of remainders, this says m must divide the integer $br - ar = (b-a)r$. Since m is prime, Euclid's Lemma implies that m must divide either $b-a$ or r. But the largest $b-a$ could be is $m-2$ and $b-a > 0$, so m cannot divide $b-a$. Similarly, m cannot divide r since $1 \le r \le m-1$. This gives a contradiction. Hence the $(t \cdot r)\ R\ m$ must all be distinct (i.e. different) as t ranges from 1 to $m-1$.

(B) (5) What does the fact from part A show about the remainders $r^{m-1}\ R\ m$? Explain.

It implies that these are all equal to 1: $r^{m-1}\ R\ m = 1$ for all r with $1 \le r \le m-1$. The reason is that we can multiply together all the numbers $(t \cdot r)\ R\ m$ and take the remainder on division by m. On the one hand, the fact that those remainders are all distinct implies that the product is equal to $(m-1)!\ R\ m$. But then we also have that

$$(m-1)!\ R\ m = r^{m-1}(m-1)!\ R\ m$$

which implies that m divides $(m-1)!(1 - r^{m-1})$. As in the first part of the problem, since m is prime, we can apply Euclid's lemma to see that m must divide either $(m-1)!$ or $1 - r^{m-1}$. The first is impossible because all of the factors in $(m-1)!$ are strictly less than m. So m must divide $1 - r^{m-1}$. But that shows that $r^{m-1}\ R\ m = 1$ as claimed.

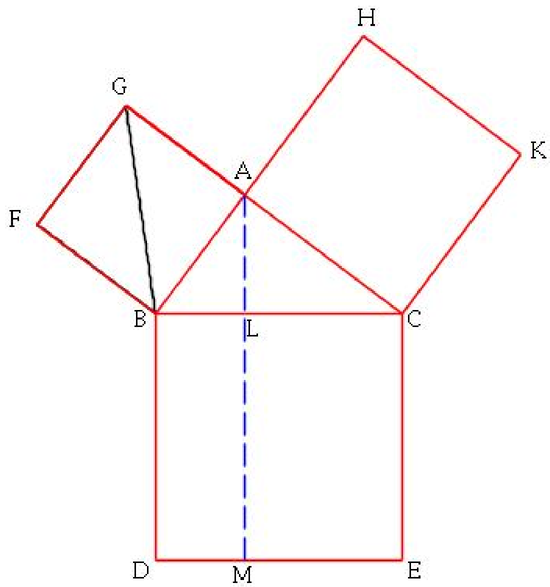

Figure 2: Figure for Proposition 47, Book I

V. Proposition 47 in Book I of the *Elements* is a form of the Pythagorean theorem, illustrated by the figure above. Use the labeling here in your answers to all parts. Euclid's form of the statement is that the area of the square on the hypotenuse of the right triangle ΔABC is equal to the sum of the areas of the squares on the sides.

(A) (5) How is the dotted line AM in the figure constructed?

Solution: It's constructed to pass through A and be *parallel* to the line containing BD. The construction for that is given in a previous proposition (Proposition 31, to be exact).

(B) (5) In the first part of the proof, Euclid shows that ΔGBF has the same area as what other triangle in the figure? Why does that follow?

Solution: ΔGBF has the same area as ΔCBF. This follows because those two triangles have the same base and are in the same parallels (Proposition 37). Euclid establishes that CG and FB lie on parallel lines by considering the alternate interior angles for the transversal line containing AB.

(C) (5) The second part of the proof consists of showing that ΔFBC and ΔABD are congruent. How does that follow? (Show that is true using one of the triangle congruence results proved before in Book I.)

Solution: We have $FB = AB$ since they are two sides of the same square. Similarly $BC = BD$ since they are two sides of the same square. Finally, $\angle FBC = \angle FBA + \angle ABC = \angle ABC + \angle CBD = \angle ABD$, where the middle equality uses the facts that

$\angle FBA$ and $\angle CBD$ are both right angles and Postulate 4. Then ΔFBC and ΔABD are congruent by the SAS congruence criterion (Proposition 4).

(D) (5) How does Euclid conclude that $ABFG$ and $BLMD$ have the same area? And how does he conclude the proof?

Solution: First he shows that ΔABD and ΔBDM have the same area using Proposition 37 again. Then Common Notion 1 says ΔGBF and ΔBDM have the same area. But ΔGBF has half the area of the square $ABFG$ and ΔBDM has half the area of the rectangle $BLMD$. So $ABFG$ and $BLMD$ also have the same area (Common Notion 2). Euclid concludes the proof by saying that a similar argument shows the area of the other square $ACKH$ is equal to the area of the rectangle $LMEC$. Then adding we get that the area of the square $BDEC$ is equal to the sum of the areas of the squares $ABFG$ and $ACKH$.

VI. Essay. (35)

Option A: Oliver Heaviside, 1850-1925, an English engineer, applied mathematician, and physicist, once wrote the following about the role of Euclid in mathematical education in his time in England: "As to the need of improvement there can be no question whilst the reign of Euclid continues. My own idea of a useful course is to begin with arithmetic, and then not Euclid but algebra. Next, not Euclid, but practical geometry, solid as well as plane; not demonstration, but to make acquaintance. Then not Euclid, but elementary vectors, conjoined with algebra, and applied to geometry ... Elementary calculus should go on simultaneously ... . Euclid might be an extra course for learned men, like Homer. But Euclid for children is barbarous." On the other hand, about 5 years ago, Peter Rudman, a contemporary physicist, wrote this: "High school mathematics education today, ... , all too often neglects the derivations where mathematics is learned and emphasizes memorizing the equations that provide quick solutions in the standardized tests but that are then rapidly forgotten ... ." What aspects of mathematics does each of these authors seem to value most highly and think students should learn? How does what each of them says relate to the ideas of G. H. Hardy in *A Mathematician's Apology*? Why might Heaviside say that teaching Euclid to children is "barbarous?" Was your high school mathematics more or less like what Heaviside is recommending? Was your experience like that Rudman describes? Do you think that emphasizing proofs more would make mathematics more interesting for more people? Or is that too much to hope for?

Model response: Heaviside seems to be stressing the parts of mathematics that are valuable for practical applications: arithmetic, "practical geometry," vectors, algebra, calculus. He specifically says that in geometry he wants students to "make acquaintance," i.e. to learn the basic language, probably formulas for areas, volumes, etc. but *not to focus on "demonstration"* (i.e. proofs). He says the study of Euclid might come later for those who will go on to specialize in mathematics, but that Euclid is inappropriate for beginning students. Rudman, on the other hand, thinks that just learning facts and memorizing formulas is not worthwhile because it does not promote real learning about logical reasoning and problem solving. He sees the derivations (the proofs) as the place where real mathematics happens and real learning takes place. Rudman's point of view is much closer to Hardy's than Heaviside's is. In fact it's probably just the parts of mathematics that Heaviside is really committed to that Hardy would find "ugly" and utilitarian. I think the most likely reason for Heaviside's idea that teaching Euclid to children is "barbarous" is probably a combination of several things. First, if Euclid is taught poorly, it can end up being more about arcane and meaningless definitions of not very useful terms, about following particular formats for writing down proofs, about learning hard-seeming proofs for "obvious" facts, and so forth. If that's the way students are taught, it can be just as dull and stifling as the worst of "teaching to the test" (what Rudman is concerned about). [For the last section, I'll just be looking for whether you can articulate reasonable ideas about the best way of teaching mathematics and about your own experience in previous courses and –no "right" or "wrong" answers there.]

Option B: (If you are choosing this option, ask Prof. Little for a copy of the poem to consult while you are writing.) The poem *Ithaka* by C. Cavafy clearly draws on themes from the *Odyssey*, but does it just retell parts of Homer's story, or does it end up making something quite different of them? In particular, is the return of Odysseus the main point here? If not, what is the main point? Why doesn't Cavafy mention Telemachus or Penelope? Finally, how do you think what Cavafy is saying here relates to the CHQ theme (especially the "how then shall we live?" part)?

Model response: In his poem *Ithaka*, Cavafy is drawing on episodes and situations from the *Odyssey* (the Laistrygonians, the Cyclops, angry Poseidon, etc.) But Cavafy is definitely making his own use of those characters and images and *not* just retelling the story Homer told. For Homer, in a sense, Odysseus' return to Ithaca from the Trojan War is the whole point of the story (especially the way his ancient Greek listeners and readers would have understood it). Odysseus' journey is full of obstacles and perils (including the Laistrygonians, the Cyclops, angry Poseidon, etc.) The story of Telemachus' parallel journey to manhood is important too, as is Penelope's strength and cleverness in keeping the suitors at bay to make it possible for Odysseus' to regain his kingdom. But Cavafy is using those themes in a much more metaphorical and psychological sense. He is essentially offering the reader his advice on how to live a good and meaningful life. He says, in effect: First and foremost, value the journey of life itself over any ultimate destination. Hope the journey lasts long and that you find great pleasure in all the wonderful things life on earth has to offer: "sensual perfumes," "fine things," "stores of knowledge." Keep the destination in mind but don't let it hurry you. Don't be too afraid of the perils (the Laistrygonians, the Cyclops, angry Poseidon, etc.) because, in some ways, the worst perils in life can be the ones we make for ourselves ("you won't encounter them unless you bring them along inside your soul, unless your soul sets them up in front of you"). And when you come to your final destination, hope that you have become so wise and full of experience through your journey that, even if the destination ends up seeming disappointing, you will understand what the purpose of your journey has been and you will arrive at that destination satisfied with what you have done in your life.

This ties in very closely with the "how then shall we live?" aspect of our CHQ theme, of course. The poem is essentially one person's answer to that question. One might argue that this answer is somewhat hedonistic (focused on pleasure) and selfish. It doesn't say much about what we might think of as our responsibility to others, about how we might try to do good in the world or to make a difference. But it's one sort of answer, or maybe one part of an answer.

Finally, as I have been hinting above, it is very plausible to read Cavafy's poem as saying that the *ultimate destination*, the metaphorical *Ithaka*, is actually *death* and that that's why savoring the journey (life itself) is so important for human beings. But there are other possibilities too – one could also read Cavafy's *Ithaka* as being any one of the big destinations we might set up for ourselves in our lives – graduating from college, marriage, children, retirement at the end of a career, etc. As is true for all really good poetry, there are multiple layers and meanings here.