College of the Holy Cross
MONT 104Q – Mathematical Journeys
Final Exam – December 18, 2015

Your Name: ______________________________

Instructions: Do all work on the test pages. Place your answers in the space provided for the problem. Use the back of the preceding page if you need more space for scratch work. The numbers next to each part of the questions are their point values. There are 100 total points on this exam.

Please do not write in the space below

Problem	Points/Poss
I	/ 10
II	/ 15
III	/ 10
IV	/ 10
V	/ 20
VI	/ 35
Total	/ 100

HAVE A PEACEFUL AND ENJOYABLE HOLIDAY SEASON!

I. G. H. Hardy included two proofs from Euclid's *Elements* as prime examples of beautiful, serious mathematics in his book *A Mathematician's Apology*. The first was the proof that there are infinitely many prime numbers.

A) (5) Prove this result using Euclid's method.

B) (5) What is the name of the method of proof that Euclid (and you) used here? Describe briefly how that method works. Hardy say this is "a far finer gambit than any chess gambit: a chess player may offer the sacrifice of a pawn or even a piece, but a mathematician offers *the game*?" What does he mean by that?

II.

(A) (10) Give the statement and proof of Proposition 29 in Book I of Euclid's *Elements*.

(B) (5) How does this proposition relate to Proposition 27? What is special about the place of Proposition 29 in Book I? Explain briefly.

III. (10) What does it mean to say that a polyhedron is *convex*? What is true about the number of vertices V, the number of edges E, and the number of faces F in that case?

IV. In our proof of the "power theorem" about the remainders r^{m-1} R m, where m is prime and $1 \leq r \leq m-1$,

(A) (5) The first step was to show that for each fixed r as above, the remainders r R m, $(2 \cdot r)$ R m, ... , $((m-1) \cdot r)$ R m were all distinct. Show this is true. You may assume the statement of Euclid's lemma about primes: if a prime m divides a product of integers $c \cdot d$, then m divides c or m divides d.

(B) (5) What does the fact from part A show about the remainders r^{m-1} R m? Explain.

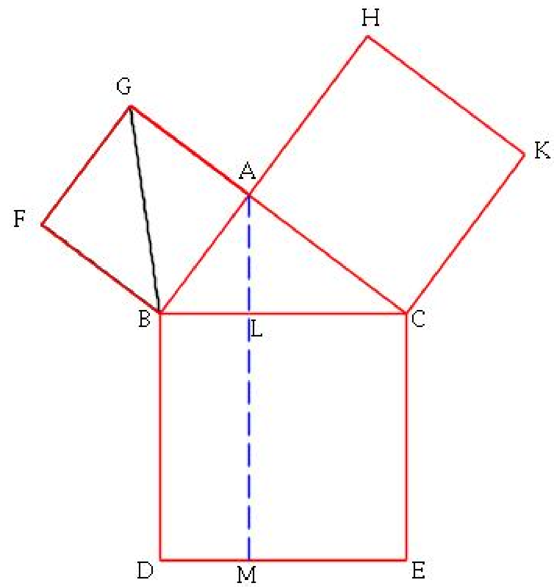

Figure 1: Figure for Proposition 47, Book I

V. Proposition 47 in Book I of the *Elements* is a form of the Pythagorean theorem, illustrated by the figure above. Use the labeling here in your answers to all parts. Euclid's form of the statement is that the area of the square on the hypotenuse of the right triangle ΔABC is equal to the sum of the areas of the squares on the sides.

(A) (5) How is the dotted line AM in the figure constructed?

(B) (5) In the first part of the proof, Euclid shows that ΔGBF has the same area as what other triangle in the figure? Why does that follow?

(C) (5) The second part of the proof consists of showing that ΔFBC and ΔABD are congruent. How does that follow? (Show that is true using one of the triangle congruence results proved before in Book I.)

(D) (5) How does Euclid conclude that $ABFG$ and $BLMD$ have the same area? And how does he conclude the proof?

VI. Essay. (35)

Option A: Oliver Heaviside, 1850-1925, an English engineer, applied mathematician, and physicist, once wrote the following about the role of Euclid in mathematical education in his time in England: "As to the need of improvement there can be no question whilst the reign of Euclid continues. My own idea of a useful course is to begin with arithmetic, and then not Euclid but algebra. Next, not Euclid, but practical geometry, solid as well as plane; not demonstration, but to make acquaintance. Then not Euclid, but elementary vectors, conjoined with algebra, and applied to geometry ... Elementary calculus should go on simultaneously ... . Euclid might be an extra course for learned men, like Homer. But Euclid for children is barbarous." On the other hand, about 5 years ago, Peter Rudman, a contemporary physicist, wrote this: "High school mathematics education today, ... , all too often neglects the derivations where mathematics is learned and emphasizes memorizing the equations that provide quick solutions in the standardized tests but that are then rapidly forgotten ... ." What aspects of mathematics does each of these authors seem to value most highly and think students should learn? How does what each of them says relate to the ideas of G. H. Hardy in *A Mathematician's Apology*? Why might Heaviside say that teaching Euclid to children is "barbarous?" Was your high school mathematics more or less like what Heaviside is recommending? Was your experience like that Rudman describes? Do you think that emphasizing proofs more would make mathematics more interesting for more people? Or is that too much to hope for?

Option B: (If you are choosing this option, ask Prof. Little for a copy of the poem to consult while you are writing.) The poem *Ithaka* by C. Cavafy clearly draws on themes from the *Odyssey*, but does it just retell parts of Homer's story, or does it end up making something quite different of them? In particular, is the return of Odysseus the main point here? If not, what is the main point? Why doesn't Cavafy mention Telemachus or Penelope? Finally, how do you think what Cavafy is saying here relates to the CHQ theme (especially the "how then shall we live?" part)?