I. Identifications. For any 5 of the following historical figures, give the approximate historical period in which they lived, where they were active, and the main contributions they made that we discussed. (If you submit answers for more than 5, only the best 5 will be counted.)

(A) (10) Eratosthenes – about 276 - 195 BCE, active in Alexandria. He made a surprisingly accurate estimate of the radius of the Earth based on measurements with a gnomon (sundial). He also developed a method for finding all prime numbers less than or equal to given number (the "sieve method").

(B) (10) Claudius Ptolemy – about 90 - 168 CE, active in Alexandria. He made some contributions to geometry (in particular a theorem about the relation of the edges and the diagonals in cyclic quadrilaterals). He wrote on astronomy and developed the largest geographical database known from the ancient world, allowing for construction of maps of the known world at the time.

(C) (10) Muhmammad ibn Musa al-Khwarizmi – about 780 - 850 CE, active in Baghdad. He wrote a treatise on algebraic techniques (the word "algebra" comes from a corruption of the Arabic in the title). He also wrote a text about the use of Hindu-Arabic numerals (our base-10 positional system) that was very influential

(D) (10) Thabit ibn Qurra – about 826 - 901 CE, active in Baghdad. He was active in the House of Wisdom set up by the Caliph al-Mansour and participated in the translation of many Greek mathematical texts into Arabic. He extended the results of Archimedes' *Quadrature of the Parabola*, made a critical reexamination of the basis of Euclidean geometry (especially the role of Postulate 5) and extended Euclid's work on perfect numbers to give a similar result for amicable pairs.

(E) (10) Leonardo Pisano ("Fibonacci") – about 1170 - 1250 CE, active in Italy (various cities). He wrote a book introducing the Hindu-Arabic numerals to scholars in western Europe and made various other algebraic and numerical contributions, including the definition of the well-known Fibonacci sequence.

(F) (10) Filippo Brunelleschi – about 1377 - 1446 CE, active in Florence. He was an architect and artist whose best-known work was the dome of the cathedral in Florence (still the largest masonry dome in existence in the world). He studied geometry deeply for use in design, and also for basis of linear perspective in painting (methods for reproducing what the eye sees in a 3D scene on a flat 2D canvas or piece of paper).

II.

(A) (10) What are the 5 Common Notions (Axioms) and 5 Postulates that appear at the beginning of Book I of Euclid's *Elements*?

Answer: The Five Common Notions are:

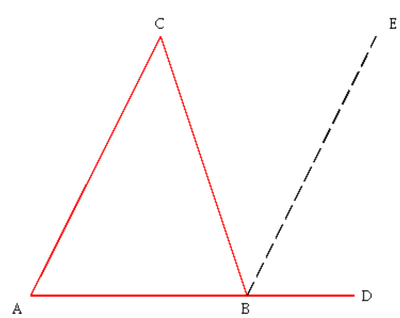
(a) Things equal to the same thing are equal to one another
(b) If equals are added to equals, the results are equal
(c) If equals are subtracted from equals, the remainders are equal
(d) Things that coincide with one another are equal to one another
(e) The whole is greater than the part

The Five Postulates are:

(a) (It is possible) to connect any two points with a straight line
(b) (It is possible) to extend a line indefinitely in both directions
(c) (It is possible) to draw a circle with any given center and any given radius
(d) All right angles are equal
(e) If two lines falling on a third line make angles summing to less than two right angles on one side, then the two lines, if extended, will meet on that side of the third line.

(B) (15) Proposition 32 in Book I deals with the sum of the angles in a triangle. What is the exact statement that Euclid proves? Outline the construction and proof that establishes this Proposition. (You don't need to quote earlier propositions by number, but do give a reason for each of your assertions.)

Answer: Proposition 32 says that (a) each exterior angle formed by extending a side of a triangle is equal to the sum of the two opposite interior angles, and (b) the sum of the interior angles in a triangle is equal to 2 right angles (or a straight, 180-degree, angle in our language). The proof is based on this construction:



Extend the side AB to D and construct a parallel BE to AC through the point B (using Proposition 31 – OK just to say this construction had been shown previously). Then in the resulting figure $\angle CAB = \angle EBD$ since those are corresponding angles for the two parallels and the transversal. Similarly, $\angle EBC = \angle ACB$ since those are alternate interior angles. From this, we get part (a) immediately from Common Notion 2. This also shows (again by Common Notion 2) part (b): that the angle $\angle DBA$ is equal to the sum of the angles in the triangle. But $\angle DBA$ is equal to 2 right angles since D, B, A are all on the same line (previous proposition). QED

Choose either this problem (III) or III' on the next page. (If you submit solutions for both, only the better of the two will be used.)

(A) (15) What is true about the sum of the angles in a triangle under the alternate *hyperbolic* form (H) of Postulate 5? Outline the proof, assuming the needed properties of Saccheri quadrilaterals.

Answer: Under the assumption (H), the sum of the angles in any triangle is strictly less than 2 right angles. This is proved as follows:

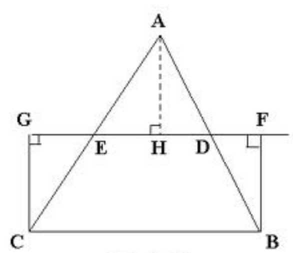


Fig. 9.24

E is constructed to be the midpoint of AC, so $AE = CE$. Similarly D is constructed to be the midpoint of AB, so $AD = DB$. Then we extend the line ED and drop perpendiculars from A, B, and C. By the AAS side criterion (using equality of vertical angles and Postulate 4), ΔCGE and ΔAHE are congruent, as are ΔAHD and ΔBFD. This shows that $CG = HA = BF$, so $CBFG$ is a Saccheri quadrilateral by definition. We know, however that the sum of the "summit angles" in such a quadrilateral is less than two right angles. Here the "summit angles" are $\angle GCB$ and $\angle FBC$. But then by the congruences proved before, the sum of the angles in the triangle ΔABC is equal $\angle GCB + \angle FBC$, so the angle sum is less than two right angles.

(B) (10) Was the result in part A the contradiction that Girolamo Saccheri, S. J. was seeking under the assumption (H)? Explain. What did Janos Bolyai and Nikolai Ivanovich Lobachevsky eventually show about the search for a contradiction starting from (H)?

Answer: No, this is not a contradiction because the angle sum theorem from Euclid's Proposition 32 (see problem II) is only valid under his Postulate 5. (The equality of the corresponding angles and the alternate interior angles formed by a transversal on two parallel lines from Proposition 29 is proved using Postulate 5.) Bolyai and Lobachevsky (and Gauss too) eventually realized that no contradiction would result from (H). There is another equally valid geometric theory here, the one we now call "hyperbolic geometry."

III'. (If you submit solutions for both this problem and III on the last page, only the better of the two will be used.)

(A) (15) The following problem appears in a text by the Indian mathematician Mahavira (about 850 C.E.). Solve it using our modern notation and algebraic methods: Of a collection of mangoes, the king took $1/6$, the queen $1/5$ of the remainder, the chief princes $1/4, 1/3, 1/2$ of the successive remainders, and the youngest child took the remaining 3 mangoes. Oh you who are clever in miscellaneous problems on fractions, give the measure (number) of the original collection of mangoes.

Answer: The problem can be solved either by "breaking down" or "building up." If we build up from the 3 mangoes the youngest child got, then the last chief prince took $1/2$, so there were 6 mangoes left before he made his choice. Similarly the second chief prince took $1/3$, so there were 9 mangoes left before his choice. Working back in this way, we can see the original pile contained 18 mangoes and each of the six people involved took equal shares – 3 mangoes each.

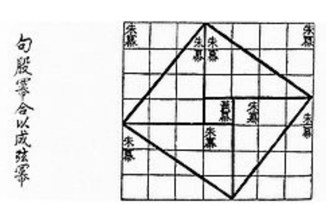
(B) (10) What was probably the single largest contribution of ancient Indian mathematics to the way we do mathematics today? How and when did this contribution enter our mathematical tradition?

Answer: It was pretty definitely the use of the base-10 positional number system containing digits $0, 1, 2, \ldots, 9$ and methods for computing with that system. Knowledge of this system reached Europe from India by way of the Islamic world in the middle ages. The book by al-Khwarizmi popularized this system in the Islamic world, and European mathematicians such as Fibonacci learned it from Islamic sources.

IV.

A) (15) Draw the diagram for the Chinese "*go gou* theorem" (with a general right triangle, sides a, b and hypotenuse c) and show how it provides an algebraic/geometric proof of the general "Pythagorean theorem."

Answer: Here is the original Chinese figure we saw in class:



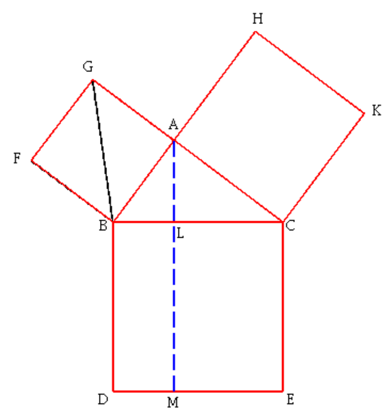
If the longer sides of the triangles are all b and the shorter sides are all a, then the four triangles and the smaller central square fit together to make a large square with side c, the hypotenuse of the triangles. By adding areas, we see

$$c^2 = 4 \cdot \frac{ab}{2} + (b-a)^2 = 2ab + b^2 - 2ab + a^2 = a^2 + b^2$$

This shows the algebraic form of the "Pythagorean theorem."

B) (10) How is this proof different from the one given by Euclid in Proposition 47 of Book I of the *Elements*?

Answer: It is mainly different in that Euclid's proof uses no algebra at all. Euclid shows that in the figure



The area of the small square on the left is equal to the area of the rectangle to the left of the blue line in the large square, and similarly, the area of the medium square on the right is equal to the area of the rectangle to the right of the blue line in the large square. So "the square on the hypotenuse equals the sum of the squares on the sides" in terms of area. This is based entirely on comparisons between areas of various triangles and parallelograms proved with no algebra.

V.

A) (15) What are *perfect numbers*? What are *amicable pairs of numbers*? Give an example of a perfect number and an amicable pair and explain how they satisfy the definitions.

Answer: A perfect number is a positive integer that is equal to the sum of its proper integer divisors. For instance, $28 = 1 + 2 + 4 + 7 + 14$ is perfect. An amicable pair of numbers is a pair m, n of positive integers with the property that the sum of the proper divisors of m is equal to n, and vice versa the sum of the proper divisors of n is equal to m. The pair $m = 220$, $n = 284$ is an amicable pair since 220 has proper divisors $1, 2, 4, 5, 10, 20, 11, 22, 44, 55, 110$ and these add up to 284, while 284 has proper divisors $1, 2, 4, 71, 142$, and these add up to 220.

B) (10) How does the historical development of knowledge about these number-theoretical ideas tie together the Greeks, the medieval Islamic mathematicians, and mathematics in the contemporary world?

Answer: The idea of perfect numbers and amicable pairs can be traced back to the Pythagoreans and their number symbolism. Euclid studied perfect numbers in Book X of the *Elements*, showing that if p is prime and $2^p - 1$ is also prime, then $2^{p-1}(2^p - 1)$ is an even perfect number. (We get can $28 = 4 \cdot 7 = 2^{3-1}(2^3 - 1)$ this way.) The medieval Islamic mathematicians learned about these ideas from their study of Euclid and other Greek texts, and extended some of them – for instance the theorem of Thabit ibn-Qurra on amicable pairs. These ideas lead to many questions whose answers are unknown even today and which are connected to research going on in contemporary mathematics – Are there any odd perfect numbers? Are there infinitely many primes p for which $2^p - 1$ is also prime? Are there infinitely many amicable pairs of numbers?

VI. Essay. (50) Here are two contrasting statements about the ultimate legacy of Greek mathematics:

- "The death of Archimedes by the hands of a Roman soldier is symbolical of a world-change of the first magnitude: the Greeks, with their love of abstract science, were superseded ... by the practical Romans. The Romans were a great race, but they were cursed with the sterility which waits upon practicality. They did not improve upon the knowledge of their forefathers, and all their advances were confined to the minor technical details of engineering. They were not dreamers enough to arrive at new points of view ... No Roman lost his life because he was absorbed in the contemplation of a mathematical diagram." (Alfred North Whitehead – 20th century philosopher)
- "There is no denying that the Greek approach to mathematics produced remarkable results, but it also hampered the subsequent development of the subject. ... The Greek preoccupation with geometry ... was a serious constraint. Great minds such as Pythagoras, Euclid, and Apollonius spent much of their time creating what were essentially abstract idealized constructs; how they arrived at a conclusion was in some way more important than any practical significance." (George G. Joseph, *The Crest of the Peacock*)

Begin by summarizing and explaining your understanding of each statement. Then address the following questions: Have we seen examples in this class where practicality was *not sterile*, where concern with applications actually enriched the history of mathematics and made further progress possible? Have we seen other examples where the "abstract idealized constructs" of pure mathematics led to major advances? Why do you suppose that Joseph doesn't mention Archimedes? Does that omission weaken his point?

A possible response: Whitehead is saying that if a mathematical culture concentrates only (or maybe too much) on practical applications (as he claims the Romans did), then their mathematics can be *sterile*, in the sense that they will not be ready to develop new ideas as needed. He compares the Romans to the Greeks and finds the Greeks to be more imaginative "dreamers" who ultimately contributed much more to the development of mathematics than the Romans did.

Joseph is saying, on the other hand, that if a mathematical culture focuses only (or too much) on theoretical mathematics, with its characteristic preoccupations concerning methods of proof ("how they arrived at results"), then their mathematics can also "miss out" by not developing practical applications. He singles out the Greeks as a key example of this, saying that their theoretical orientation (for instance, the lack of notions of numerical measures of lengths or angles) may in fact have constrained or limited the development of mathematics.

In a sense, combining the underlying points here, one could argue that the ideal mix is a healthy combination of pure and applied mathematics, each reinforcing the other. But Whitehead is saying the Romans went too far in one direction and the Greeks were more productive, while Joseph is saying the Greeks went too far in the other direction.

As this applies to Greek mathematics in particular, one example that shows how practicality can lead to great mathematical advances is, ironically considering Whitehead's point of view, the case of Archimedes himself. One could argue that some of Archimedes' most profound and forward-looking pure mathematical work, the *Quadrature of the Parabola*, was based on his understanding of the very practical physics of balances, centers of mass, and so forth. The way the Islamic mathematicians of the middle ages combined Greek geometry with more practical Mesopotamian algebra and Indian arithmetic is another example of how (some) practicality can reinforce and enrich pure mathematics.

On the other hand, it was the "abstract idealized concepts" of pure mathematics, in particular the basis from Euclid, the work of Apollonius on conics, etc. that made it possible for Archimedes to do what he did. Similarly, later, it was the theoretical basis of Euclidean geometry that made it possible for Renaissance architects and painters to build things like the dome of the cathedral in Florence and develop realistic perspective in painting.

Mentioning Archimedes would certainly weaken Joseph's point since his work combined the theoretical and the practical in such fruitful ways. But Joseph would also probably say that Archimedes belongs to the Hellenistic period and marks the start of a different phase of Greek mathematics. He would say he's thinking mainly of the earlier period of Pythagoras, Euclid, and Apollonius. By the time we get to the time of Heron, Claudius Ptolemy, etc. Greek mathematics has acquired a very practical side in addition to the theoretical preoccupations. But Archimedes is actually earlier than that too, so it's subtle!