Math in the Mountains Toric Varieties Tutorial
July 29 - 31, 2013

Exercises on Toric Varieties

1. Show that the rational quartic in $\mathbf{P}^3$ corresponding to $\mathcal{A} = \{(4,0),(3,1),(1,3),(0,4)\}$ is not projectively normal (Harshorne, I.3.18)
2. Show that the tetrahedron $P = \mathrm{conv}\{0, e_1, e_2, e_1 + e_2 + 2e_3\} \subset \mathbf{R}^3$ is not normal. (A lattice polytope $P \subset M_{\mathbf{R}}$ is normal if $(kP \cap M) + (\ell P \cap M) = (k+\ell)P \cap M$ for all $k, \ell \in \mathbf{N}$.)
3. Show that the complete fan Σ in $N_{\mathbf{R}} \simeq \mathbf{R}^2$ with rays $\mathbf{R}_{\geq 0}e_1$, $\mathbf{R}_{\geq 0}e_2$, and $\mathbf{R}_{\geq 0}(-e_1 - e_2)$ gives $X_\Sigma = \mathbf{P}^2$.
4. Identify the toric surface given by the complete fans with rays $\pm\mathbf{R}_{\geq 0}e_1$, $\pm\mathbf{R}_{\geq 0}e_2$ (four 2-dimensional cones)
5. Identify the toric surface given by the complete fans with rays $\mathbf{R}_{\geq 0}e_1$, $\pm\mathbf{R}_{\geq 0}e_2$, and $\pm\mathbf{R}_{\geq 0}(-e_1 + ae_2)$ where $a \geq 1$ (four 2-dimensional cones)
6. Show using the toric description that $\mathrm{Cl}(\mathbf{P}^1) = \mathbf{Z}$.
7. Complete the proof that we have an exact sequence for the class group of X_Σ:

$$M \to \mathbf{Z}^{|\Sigma(1)|} \to \mathrm{Cl}(X_\Sigma) \to 0,$$

   where the first map is given by the integer matrix whose rows are the primitive generators of the rays of Σ.
8. (Toric surfaces, Riemann-Roch, etc.) The Riemann-Roch theorem on a smooth algebraic surface X can be written as

$$\chi(\mathcal{O}_X(D)) = \frac{D(D - K_X)}{2} + \chi(\mathcal{O}_X), \tag{1}$$

   where D is a divisor on X, K_X is the canonical divisor, and

$$\chi(\mathcal{F}) = h^0(\mathcal{F}) - h^1(\mathcal{F}) + h^2(\mathcal{F})$$

   is the Euler characteristic (there would be additional terms if $\dim X > 2$).

   a. Given a lattice polytope P in $\mathbf{Z}^2$, show how to use the facet description of P to find D and Σ such that a basis for $H^0(\mathcal{O}_{X_\Sigma}(D))$ is given by $P \cap \mathbf{Z}^2$.
   b. Show that in this case $\chi(\mathcal{O}_{X_\Sigma}(D)) = |P \cap \mathbf{Z}^2|$ and $\chi(\mathcal{O}_{X_\Sigma}) = 1$ (Hint: Demazure + support function.)
   c. Replace D with nD in the formula (1) above to get

$$h^0(\mathcal{O}_{X_\Sigma}(nD)) = \frac{D^2}{2}n^2 - \frac{DK}{2}n + 1$$

   d. Pick's formula for a lattice polygon implies

$$|nP \cap \mathbf{Z}^2| = \mathrm{Area}(P)n^2 + \frac{|\partial P \cap \mathbf{Z}^2|}{2}n + 1.$$

Check this for the triangle $P = \mathrm{conv}\{(0,0),(2,0),(0,2)\}$.

e. Conclude: We can compute D^2 and DK from purely combinatorial data from P for a toric surface. In fact $K_{X_\Sigma} = -\sum_{\rho\in\Sigma(1)} D_\rho$ when X_Σ is smooth.

Exercises on Algebraic Statistics Topics

1. Give an integer matrix $\mathcal{A}$ such that the corresponding toric model coincides with the 3×3 independence model. Generalize to obtain a matrix corresponding to the $k\times\ell$ independence model for all $k,\ell \geq 2$.

2. For the Jukes-Cantor model,
   a. Compute $P(ACG)$, $P(AAC)$, $P(ACA)$ and $P(CAA)$. Explain where the p_{dis}, p_{ij} polynomials come from and why every component of the full Jukes-Cantor model can be expressed using just these 5 distinct polynomials.
   b. Verify that $p_{123} + p_{dis} + p_{12} + p_{13} + p_{23} = 1$.
   c. Show that

$$\begin{aligned}
q_{111} &= (\theta_1 - \pi_1)(\theta_2 - \pi_2)(\theta_3 - \pi_3)\\
q_{110} &= (\theta_1 - \pi_1)(\theta_2 - \pi_2)(\theta_3 + 3\pi_3)\\
q_{101} &= (\theta_1 - \pi_1)(\theta_2 + 3\pi_2)(\theta_3 - \pi_3)\\
q_{011} &= (\theta_1 + 3\pi_1)(\theta_2 - \pi_2)(\theta_3 - \pi_3)\\
q_{000} &= (\theta_1 + 3\pi_1)(\theta_2 + 3\pi_2)(\theta_3 + 3\pi_3)
\end{aligned}$$

   are linear combinations of p_{123}, p_{dis}, p_{ij} (and monomials in linear combinations of the original parameters).
   d. What is the corresponding toric variety for this reparametrized model?

3. Let $\mathcal{A}$ be any integer matrix with equal column sums. Let φ be the parametrization of the corresponding toric model. Write $\widehat{p}$ for $\varphi(\widehat{\theta})$, where $\widehat{\theta}$ is the MLE for θ. Show using the method of Lagrange multipliers that if L is the likelihood function for given data u, then

$$L\cdot b = L\cdot Au = \lambda\cdot A\widehat{p}$$

Exercises on Geometric Modeling Topics

1. Show that for all choices of the control points $P_i \in \mathbf{R}^2$, the Bézier cubic curve

$$b(t) = (x(t),y(t)) = \sum_{i=0}^{3} \binom{3}{i} t^i (1-t)^{3-i} P_i$$

   lies on an algebraic curve defined by an equation $F(x,y) = 0$, where F has total degree 3 in x,y. The curve $C = V(F)$ is automatically rational ($g(C) = 0$) because of this. Explain why this implies C must have a singular point (an ordinary node or cusp), possibly with coordinates in $\mathbf{C}$.

2. Some questions about toric surface patches.

   a. Determine the Krasauskas toric blending functions for the triangle

   $$\Delta_k = \text{conv}\{(0,0),(k,0),(0,k)\}$$

   and show that the corresponding surface patch is a reparametrization of the Bézier triangle patch of order k.

   b. Now consider the Krasauskas toric blending functions for the triangle

   $$\Delta = \text{conv}\{(0,0),(2,0),(1,1)\}$$

   What happens on the image of a toric surface patch for this Δ at the corner $(1,1)$?

   c. Let P be a vertex of Δ. The "corner triangle" of P is the triangle consisting of P and the nearest two lattice points on the edges adjacent to P. One of the results of [K] is that the image of P on the toric surface patch is nonsingular if and only if the corner triangle has area $1/2$ (equivalently, the two vectors from P to the nearest lattice points on the adjacent edges form a lattice basis for $\mathbf{Z}^2$). Prove this statement and identify the singularity if the corner triangle has area $\ell/2$ for $\ell > 1$.

3. Let $\mathcal{P}$ be a set of control points in $\mathbf{R}^3$. Show that the degree of the implicit equation defining an algebraic surface containing the image of $b_{\mathcal{A},w,\mathcal{P}}$ is at most

   $$2 \cdot \text{Area}(\text{conv}(\mathcal{A}))$$

   and equals this for generic choices of $\mathcal{P}$. (Hint: Use the BKK bound.)

4. What is the relation between the image of $b_{\mathcal{A},w,\mathcal{P}}$ and the image of $b_{\mathcal{A}',w,\mathcal{P}'}$ if $\mathcal{A}' \subset \mathcal{A}$ and $\mathcal{P}'$ is the subset of $\mathcal{P}$ corresponding to $\mathcal{A}'$?

5. Let Δ be a lattice polygon in the plane, let h_m be the equations of the edges using inward normals, and let $H : \Delta \to \mathbf{R}^\ell$ be the map defined by the h_i The toric blending functions come from composing this H with $\chi : \mathbf{R}^\ell \to \mathbf{R}^\ell$ defined by

   $$y \longmapsto (y_1^{h_1(m)} \cdots y_\ell^{h_\ell(m)} : m \in \mathcal{A})$$

   The toric surface patch is the composition $\pi_P \circ \chi \circ H$, where $\pi_{\mathcal{P}} : \mathbf{R}^\ell \to \mathbf{R}^3$ is the affine form of a linear projection defined by the set of control points Show that the m-component of $\chi(y)$ is just $y^a z^m$, where $a = (a_1, \ldots, a_\ell)$ comes from the constant terms, and $z = (z_j)$ where

   $$z_j = \prod_{i=1}^{\ell} y_i^{\langle \mathbf{v}_i, \mathbf{e}_j \rangle}, \quad j = 1, 2$$

6. Show that the toric suface patch for the triangle $\text{conv}\{(0,0),(k,0),(0,k)\}$ has linear precision.

7. Exactly how does the set-up for Birch's Theorem relate to the algebraic moment map for $Y_{\mathcal{A}}$?