College of the Holy Cross, Fall Semester, 2017
MATH 243, Section 2 – Midterm 1
Friday, September 29

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1. (a) (15) Construct the truth table for the statement

$$((\text{not } Q) \text{ and } (P \text{ implies } Q)) \text{ implies } (\text{not } P).$$

Solution: The truth table for ((not Q) and (P implies Q)) implies (not P) looks like this:

P	Q	((not Q)	and (P	implies Q))	implies	(not P)
T	T	F	F	T	T	F
T	F	T	F	F	T	F
F	T	F	F	T	T	T
F	F	T	T	T	T	T

(The next to last column is the truth value of the whole statement.)

(b) (5) Suppose you know that $x \in \mathbf{R}$ implies that $x^2 \geq 0$ and someone one hands you a mathematical object x for which you can define x^2, and for which x^2 is a real number with $x^2 < 0$. What can you conclude?

Solution: Let P be the statement $x \in \mathbf{R}$ and let Q be the statement $x^2 \geq 0$. By part (a), you can say that not$P$ must be true, so x is not a real number.

2. Let

$$A = \{x \in \mathbb{Z} : x = 3k + 1, \text{some } k \in \mathbb{Z}\}$$
$$B = \{x \in \mathbb{Z} : x = 4k, \text{some } k \in \mathbb{Z}\}$$
$$C = \{-9, -8, -7, -6, -5, -4, -3, -2, -1\}.$$

(a) (7) What is the set $(A \cup B) \cap C$?

Solution: $(A \cup B) \cap C$ consists of the elements in C that are also either in A or B or in both. This gives

$$(A \cup B) \cap C = \{-8, -5, -4, -2\}.$$

(b) (7) What is the set $B^c \cap C$?

Solution: The complement of B is the set of integers that are not multiples of 4. The intersection of that set with C is $\{-9, -7, -6, -5, -3, -2, -1\}$.

(c) (6) Prove or disprove: $4A \subseteq A \cap B$, where $4A = \{4x : x \in A\}$.

Solution: This is true because, for instance, if $x \in 4A$, then $x = 4\cdot(3k+1) = 12k+4$ for some $k \in \mathbb{Z}$. This says $x \in B$ directly since $3k+1 \in \mathbb{Z}$. But we also have $x = 12k+4 = 3\cdot(4k+1)+1$, so $x \in A$. Hence $x \in A \cap B$ and this shows $4A \subseteq A \cap B$.

3. Consider the statement about $x \in \mathbb{Z}$: If $x^2 \notin 1 + 4\mathbb{Z}$, then x is even.

   (a) (10) What is the contrapositive form of this implication?

   Solution: If x is odd, then $x^2 \in 1 + 4\mathbb{Z}$. (Note: "not even" is the same as odd here!)

   (b) (10) Prove the contrapositive form. What does this imply about the original statement.

   Solution: If x is odd, then $x = 2k+1$ for some $k \in \mathbb{Z}$. Hence $x^2 = (2k+1)^2 = 4k^2 + 4k + 1 = 4(k^2+k)+1$. Since k is an integer, so is $k^2 + k$. This shows $x^2 \in 1 + 4\mathbb{Z}$. This also proves the implication given in the statement of the problem because that is logically equivalent to the contrapositive form.

4. (20) Let

$$A = \{x \in \mathbb{Z} : x = -2 + 5k \text{ for some } k \in \mathbb{Z}\}$$
$$B = \{x \in \mathbb{Z} : x = 3 + 5k \text{ for some } k \in \mathbb{Z}\}.$$

Is $A = B$? Prove your assertion.

Solution: These sets are equal (as you will start to see if you list out some of their elements). Here is a proof of $A = B$:

$\subseteq$: let $x \in A$. Then $x = -2 + 5k$ for some integer k. We can rewrite this as $x = (-2+5) + 5(k-1) = 3 + 5(k-1)$. Since $k \in \mathbb{Z}$, $k - 1 \in \mathbb{Z}$ as well and this shows $x \in B$. Hence $A \subseteq B$.

$\supseteq$: Now we show the reverse inclusion. Let $x \in B$ Then $x = 3 + 5k$ for some $k \in \mathbb{Z}$. We can rearrange that as follows $x = 3 - 5 + 5(k+1) = -2 + 5(k+1)$. Since $k \in \mathbb{Z}$, $k + 1 \in \mathbb{Z}$ as well and this shows $x \in A$. Hence $B \subseteq A$.

Since $A \subseteq B$ and $B \subseteq A$, we have $A = B$.

5. (20) Prove by mathematical induction: If a, d are any constants and $n \geq 0$ is a natural number, then

$$a + (a+d) + (a+2d) + \cdots + (a+nd) = \frac{n+1}{2} \cdot (2a+nd)\,.$$

Solution: The base case is $n = 0$, and the formula in that case says $a = \frac{1}{2}(2a + 0 \cdot d) = a$, which is true. Now assume

$$a + (a + d) + (a + 2d) + \cdots + (a + kd) = \frac{k+1}{2} \cdot (2a + kd)$$

And consider the sum with $n = k + 1$. As typically happens in these proofs of summation formulas, the sum for $n = k$ consists of the first group of terms. So after substituting from the induction hypothesis, we get

$$\frac{k+1}{2} \cdot (2a + kd) + (a + (k+1)d).$$

If we separate out the a and the d terms we can see this is

$$((k+1)a + a) + \frac{k(k+1)}{2} \cdot d + (k+1)d = (k+2)a + \frac{(k+1)(k+2)}{2} \cdot d.$$

Hence by factoring out $\frac{(k+2}{2}$, this can be written in the form

$$\frac{(k+2)}{2}(2a + (k+1)d),$$

which gives what we wanted to show.