College of the Holy Cross, Fall Semester 2017
MATH 243 – Mathematical Structures, section 2
Final Exam – December 13

Your Name: ____________________________

Instructions: You have until 2:00pm to work on this exam. Please write your answers in the spaces provided on the following pages, and show work on the test itself. *For possible partial credit, even if you cannot completely solve a problem, include definitions of terms involved, partial results you can do, etc.* Use the back of the preceding page if you need more space for scratch work.

Please do not write in the space below

Problem	Points/Poss
I	/ 25
II	/ 20
III	/ 20
IV	/ 35
V	/ 30
VI	/ 30
VII	/ 40
Total	/200

Have a peaceful and joyous holiday season!

I.

A) (15) Let p, q represent any propositions. Construct the truth table for the proposition

$$(p \text{ implies } q) \text{ if and only if } ((p \text{ and not } q) \text{ implies not } p)$$

1. B) (5) Give the contrapositive form of the statement "If the product of two integers x, y is even, then x is even or y is even."

2. C) (5) Give the converse of the statement in part B.

II. All parts of this question refer to the mapping $f : \mathbb{Z} \to \mathbb{Z}$ defined by

$$f(x) = \begin{cases} 6x & \text{if } x \text{ is even} \\ 2 - x & \text{if } x \text{ is odd} \end{cases}$$

A) (10) Let $U = \{1, 2, 3, 4, 5\}$. What is $f(U) \cap \{x \in \mathbb{Z} : x > 0\}$?

B) (10) Let $V = 4\mathbb{Z}$. What is $f^{-1}(V)$ for this mapping? Describe using proper set notation.

III. (20) Give a precise statement of the Division Algorithm in $\mathbb{Z}$ and prove the Existence and Uniqueness parts.

IV.

A) (20) Use the Euclidean Algorithm to find the integer $d = \gcd(753, 156)$ and express d in the form $d = 753m + 156n$ for some integers m, n.

B) (15) Let a, b, c be integers. Prove that $\gcd(ab, c) = 1$ if and only if $\gcd(a, c) = 1$ and $\gcd(b, c) = 1$.

V. Let $f : A \to B$ be a mapping and $U, V \subseteq A$. For each statement, give a proof if the statement is true, or give a counterexample if the statement is false. (A complete counterexample consists of specific sets A, B, U, and V, a mapping f, and justification of why these data contradict the statement.)

A) (15) If f is injective and $f(U) \subseteq f(V)$, then $U \subseteq V$.

B) (15) If f is surjective and $f(U) \subseteq f(V)$, then $U \subseteq V$.

VI. All parts of this question refer to $R = \mathbb{Z}/30\mathbb{Z}$, in which the operations are addition and multiplication mod 30.

A) (20) Construct the addition and multiplication tables for the subset $T = \{[0], [6], [12], [18], [24]\}$ in R.

B) (10) Which elements of R have *multiplicative* inverses in R? Explain how you know.

VII.

A) (20) Prove by mathematical induction that

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{n\cdot(n+1)}=\frac{n}{n+1}$$

for all natural numbers $n \geq 1$.

B) (10) What is the least upper bound

$$L = \sup\left(\left\{\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{n\cdot(n+1)} \mid n \geq 1\right\}\right).$$

(This is the same set of numbers as in part A.) Prove your assertion by showing that for every real $\epsilon > 0$, there exists some n such that

$$L - \varepsilon < \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{n\cdot(n+1)} \leq L.$$

C) (10) What does your argument in part B say about the sequence

$$a_n = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{n\cdot(n+1)}?$$