MATH 243 – Mathematical Structures
Solutions for Quiz 9 – December 1, 2017

A) (15) State the Completeness Axiom ('Axiom C') for the real numbers.

Solution: The statement is – Let $A \subset \mathbb{R}$ be a nonempty a subset that is bounded above. Then A has a least upper bound $b \in \mathbb{R}$.

Note:

(1) Saying $b = \sup(A)$, the least upper bound, means first that b is an upper bound for A – that is, all $x \in A$, $x \leq b$ – and moreover,

(2) If $x \leq b'$ for all $x \in A$, then $b \leq b'$. (Intuitively, b is the smallest number that is an upper bound for A.)

(3) The assumption that $A \neq \emptyset$ is necessary because $\emptyset$ is bounded above, but *every* $b \in \mathbb{R}$ satisfies the condition $x \leq b$ for all $x \in \emptyset$ (there aren't any such x, so the condition is *vacuously true*). There is no smallest upper bound for A in that case.

B) (15) Let $A \subset \mathbf{R}$ and $2A = \{2x : x \in A\}$. Show that if $\sup(A) = c$, then $\sup(2A) = 2c$.

Solution 1: (a direct proof) First, since $\sup(A) = c$, we have that c is an upper bound for A. This implies that $x \leq c$ for all $x \in A$. Since $2 > 0$, we can multiply both sides of this inequality by 2 to yield $2x \leq 2c$ for all $x \in A$. This shows that $2c$ is an upper for $2A$ as in point (1) in the solution for part A above. Now, to show $2c$ is the least upper bound, we need to show that point (2) also holds for the bound $2c$ and the set $2A$. So let d be any other upper bound for the set $2A$. This means that $2x \leq d$ for all $x \in A$, so $x \leq \frac{d}{2}$ for all x in A. By point (2) for the upper bound c for A, this implies $c \leq \frac{d}{2}$. But since $2 > 0$, that implies $2c \leq d$. Hence $2c = \sup(2A)$.

Solution 2: The proof of point (1) is the same as before – if $x \leq c$, for all $x \in A$, then $2x \leq 2c$, so $2c$ is an upper bound for $2A$. Now we argue by contradiction for point (2). Suppose $2c$ is not the least upper bound of $2A$. That means that there is some $d < 2c$ that is also an upper bound for $2A$: $2x \leq d$ for all $2x \in 2A$. But multiplying by $1/2 > 0$ yields $x \leq \frac{d}{2}$ for all $x \in A$. This implies $\frac{d}{2}$ is an upper bound for A. However $d < 2c$ implies $\frac{d}{2} < c$ and that contradicts the assumption that c was the least upper bound of A.

Some further comments: It's tempting to say that $c < 2c$ and try to relate c and $2c$ that way. However, this is only true if $c > 0$. If $c < 0$, then in fact $2c < c$.